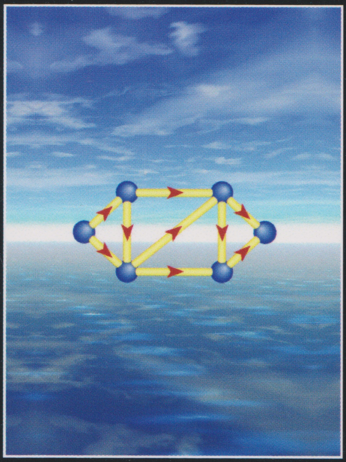
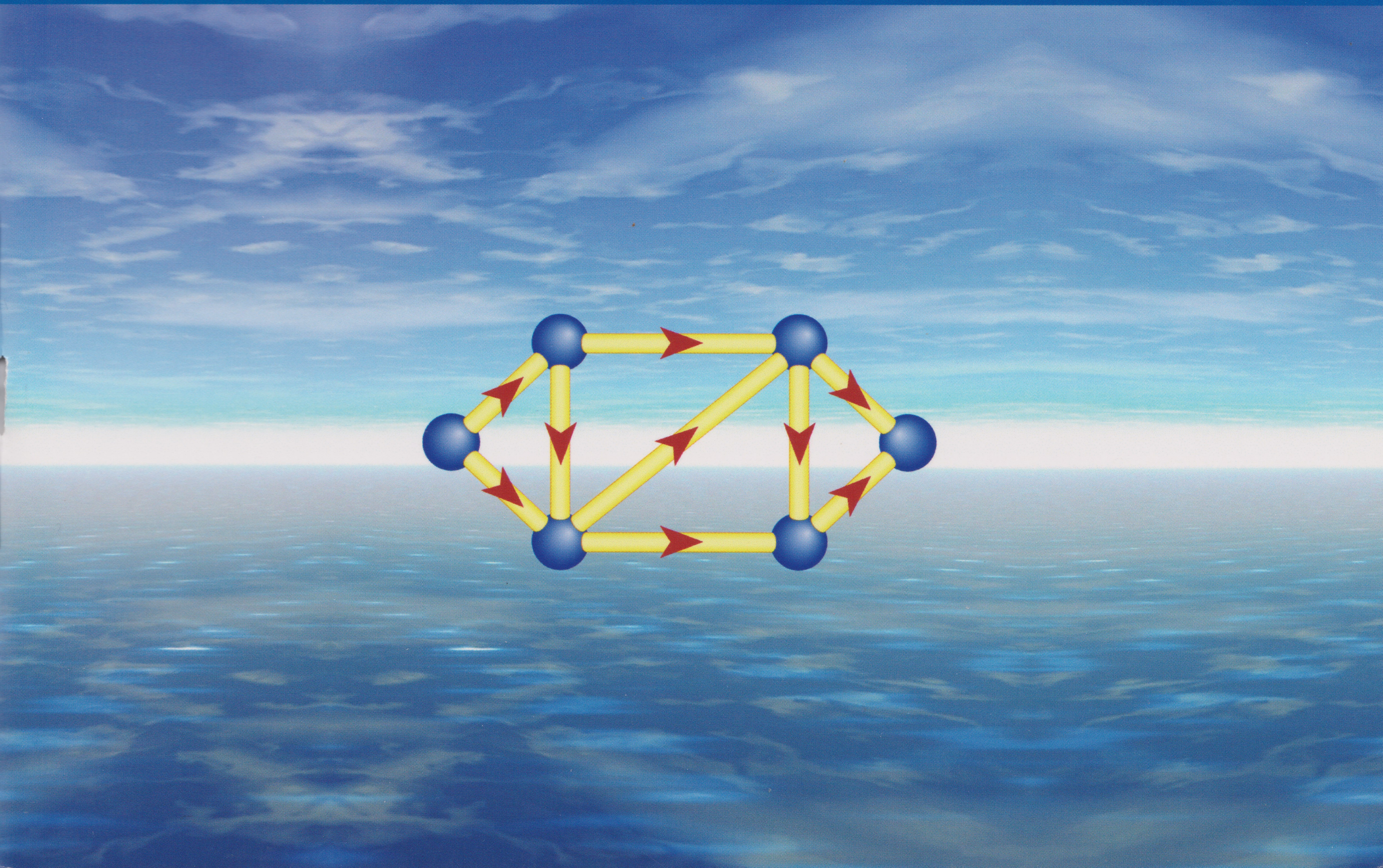
The Open University

MT365 Graphs, networks and design



Networks 4

Physical networks



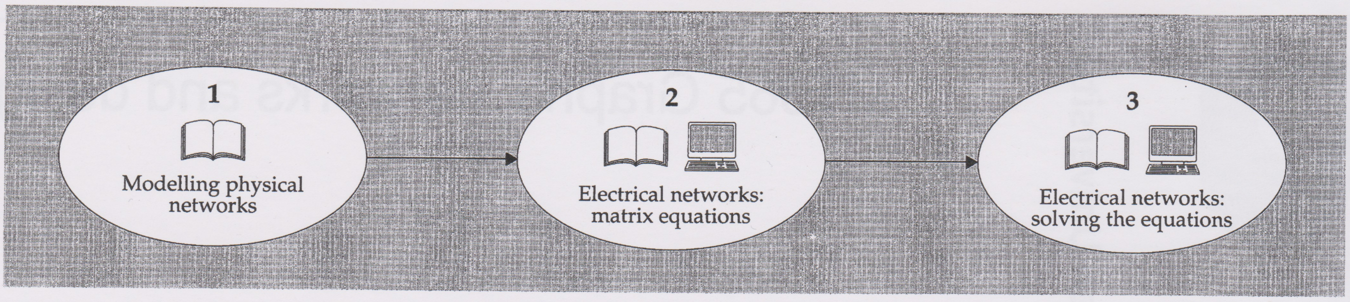
The Open University

MT365 Graphs, networks and design

Networks 4

Physical networks

Study guide



The most important sections are Sections 2 and 3. These sections involve matrix algebra, including the concept of an *inverse matrix*; if you are not familiar with this idea you may find it helpful to study the computer package **Matrix manipulation** before studying these sections.

Section 2 is the longest section and you will probably require two study sessions for this.

Unlike the earlier Networks units, this unit is not concerned with optimization problems, but is mostly concerned with a problem for which it is possible to find a *unique* solution. This can ideally be done on a computer. Thus the unit is concerned with the mathematical aspects of *setting up the problem* so that it can be solved on a computer. Once this has been done, solution of the problem is straightforward.

There is no audio-tape and no television programme associated with this unit.

The Open University, Walton Hall, Milton Keynes, MK7 6AA.

First published 1995. Reprinted 1998, 2002, 2003, 2009.

Printed by Page Bros, Norwich

ISBN 0 7492 2228 X

1.6

FSC www.fsc.org MIX Paper from responsible sources FSC® C023114

Contents

Introduction

In this unit we discuss the use of graph theory to solve certain physical network problems. Most of the discussion is in terms of electrical networks, for which the method we give can be used to find all the currents and voltages in the network. Electric currents and voltages are examples of *through* and *across variables*, and they satisfy the well-known Kirchhoff's current and voltage laws. The method we describe is very general, and can be applied to any physical network for which the through and across variables satisfy analogues of Kirchhoff's laws. Examples include hydraulic networks, mechanical systems, pneumatic networks and thermal networks.

A physical network can be represented by a graph in which the edges represent the components of the network and the vertices correspond to the terminals at which the components are connected together. Associated with each edge are the *edge-variables* — the through and across variable (currents and voltages in the case of an electrical network) which satisfy Kirchhoff-type laws. Since we need to be able to determine the directions of the through and across variables, we assign a reference direction to each edge of the graph. We call a graph with reference directions an *oriented graph*.

If the physical behaviour of the components can be modelled by equations (called the *component equations*) then these equations, together with the equations resulting from applying Kirchhoff-type laws, can be solved to find the values of all the through and across variables for the network. For networks of small size, these equations can be solved by hand, but for larger networks it is more appropriate to use a computer. We show how the equations can be formulated as matrix equations; this procedure is appropriate for solving network problems on a computer. The required number of equations resulting from Kirchhoff's laws are obtained directly from the incidence matrix of the oriented graph. This means that the only information which needs to be fed into the computer is the incidence matrix together with the component equations. This is particularly convenient, because the incidence matrix can be written down directly from the oriented graph.

In *Section 1, Modelling physical networks,* we explain what is meant by through and across variables, giving examples for different physical systems, and show how a physical system can be represented by an oriented graph.

In *Section 2, Electrical networks: matrix equations,* we show how to use a spanning tree of an oriented graph to obtain the maximum possible number of equations containing no redundant information by applying Kirchhoff's laws to the fundamental cutsets and cycles with respect to the chosen spanning tree. We then show how these equations in matrix form can be obtained directly from the incidence matrix of the oriented graph. The solution of these equations for electrical networks is discussed in Section 3, *Electrical networks: solving the network equations*; in particular, we discuss the method of *Gaussian elimination.*

1 Modelling physical networks

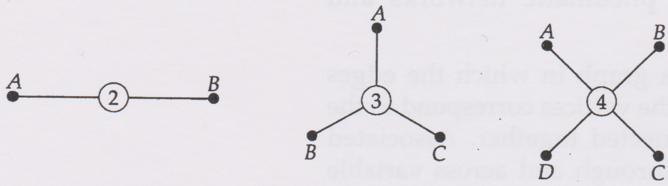
A physical network can be considered to be a number of individual components connected together. In the case of an electrical network, these components might be resistors, capacitors, transistors, transformers and so on. We first look at the representation of individual components, and the mathematical modelling of their behaviour, before going on to show how the whole connected network of components can be represented by an oriented graph.

1.1 Components and through and across variables

Components

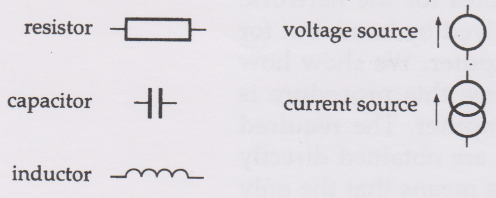
We assume that the physical systems we wish to analyse can be considered to consist of a number of interconnected *components*.

The point at which one component is connected to another is called a **terminal**. The following diagrams represent components with two, three and four terminals.

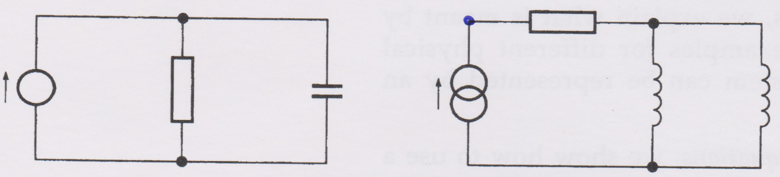


A **2-terminal component** might be a spring, a resistor, a length of wire or piping, or any other physical object which fits into a system via two connections. For example, a spring functions when both ends are appropriately connected so that it can be compressed or stretched. Similarly, a resistor functions when it is connected so that current may flow through it.

We use the following diagrams to denote electrical 2-terminal components:



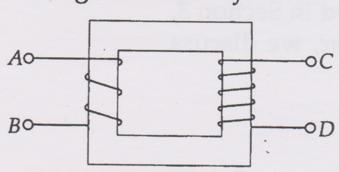
For example, diagram (a) below is the circuit diagram of an electrical circuit containing a voltage source connected to a resistor and a capacitor, and diagram (b) is the circuit diagram of a current source connected to a resistor and two inductors.



(a) (b)

A **3-terminal component** might be a *transistor*. We discuss transistors in detail later in this section.

A **4-terminal component** might be a *transformer*.



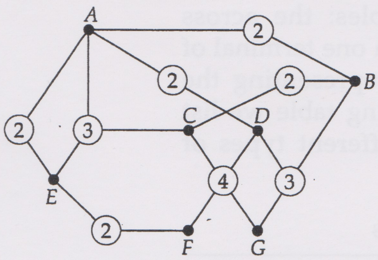
transformer

Note that for some applications one of the input terminals, *A* or *B*, may be connected to one of the output terminals, *C* or *D*, so that the transformer operates as a **3-terminal** component. See, for example, Exercise 1.2(c).

The much simplified diagram above shows a transformer in which two coils of wire, called the *primary* and *secondary*, are wound on a magnetic core so that they are in close proximity to each other but are not in electrical contact. The letters *A* and *B* denote the terminals of the primary winding, and *C* and *D* denote the terminals of the secondary winding.

We use a schematic diagram to show how the components of a particular system are connected together. For example, the following schematic

diagram represents a system comprising five 2-terminal components, two 3-terminal components and one 4-terminal component:

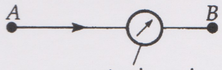


Through and across variables

In *Networks 1* we discussed flows in networks where the flow might, for example, be the flow of traffic in a road network. A flow in a network has the property that the amount of flow of a particular commodity into any vertex (other than a source or sink) is equal to the amount of flow out of it. A variable with this property is classified as a *through variable* because it represents an actual flow *through* part of a system.

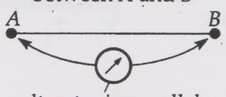
Many types of network also involve a different kind of variable called an *across variable*. To make clear the distinction between these two types of variable, let us look at a specific example, an electrical network.

In an electrical network we are concerned with the amount of current which passes through a component in the network. This current is a flow, or a *through variable*, which is measured by a meter in series with the component as in figure (a) below. But there is also an *across variable* which is the *potential difference* (or voltage) between (or *across*) any two points of the network. The potential difference can be measured by connecting a voltmeter across the two points under consideration, in parallel with the component, as in figure (b) below.



through variable

(a)

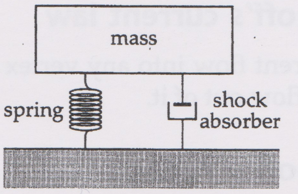


across variable

(b)

An analogous example is that of a hydraulic network, which consists of a network of pipes carrying water or some other fluid. Here the through variable is the amount of water flowing through any part of the network and the across variable is the pressure difference between any two points of the network.

Consider now a mechanical system such as a car suspension. Here we have a mass (the car) supported by a spring and a shock absorber connected onto the wheel. As far as vertical motion is concerned, the wheel is more or less 'fixed' to the road. A schematic diagram is given in the margin.

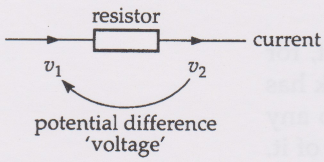
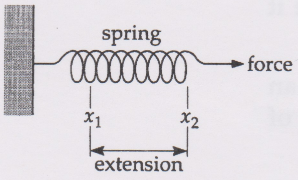
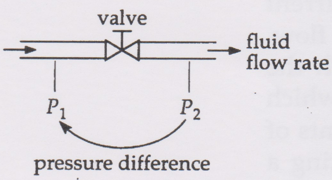
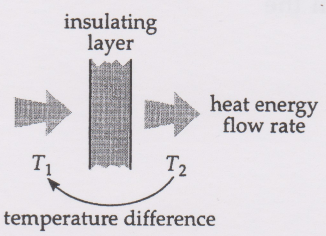


An engineer designing the suspension system is concerned both with the forces in the spring and shock absorber and with the velocity of the vertical motion of the car. The force in the spring can be measured by placing a load cell (a sophisticated version of bathroom scales) between the car and the spring, or the spring and the road; the force is then carried by the load cell as well as by the spring (just as the bathroom scales bear your weight when you are standing on them). The force is thus measured by placing a meter in series with the component, and so is a through variable. The vertical velocity of the car can be measured, using some electronic timing mechanism, by measuring the change of the distance between a point on the car and the road in a given time: the measurement

here is across the spring, from the car at one end to the ground at the other. Hence the velocity is an across variable.

In each of the above examples, there are two variables: the across variable, representing the difference of a quantity between one terminal of a component and another, and the through variable, representing the quantity flowing through the component. In the following table we list some examples of through and across variables for different types of physical system.

Table 1.1 Some examples of through and across variables

type of system	example	through variable	across variable
electrical		current	potential difference 'voltage' $v_1 - v_2$
mechanical		force	extension $x_2 - x_1$
hydraulic		fluid flow rate	pressure difference $P_1 - P_2$
thermal		heat energy flow rate	temperature difference $T_1 - T_2$

Equations relating through and across variables

For electrical networks, the currents and voltages satisfy the following two laws.

Kirchhoff's current law

The current flow into any vertex of an electrical network is equal to the current flow out of it.

Kirchhoff's voltage law

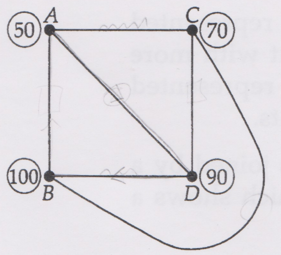
The algebraic sum of the potential differences across all the components around any circuit (cycle) in an electrical network is zero.

G. R. Kirchhoff (1824–1887)
Gustav Kirchhoff formulated his laws of electrical networks while a student at the University of Königsberg. Later he became professor of physics at Breslau, Heidelberg and Berlin.

Analogues of both these laws can be given for many types of network containing through and across variables. For example, the analogue of Kirchhoff's current law for a network carrying a flow is simply the *flow conservation law*: what flows into a vertex of a network (other than a source or sink) must flow out of it. The analogue of Kirchhoff's voltage law is the *potential difference conservation law*: the sum of the potential differences around any circuit of a network is zero.

Example 1.1

Consider the following diagram of a hydraulic network in which the number next to each vertex represents the pressure of the water or other fluid at that vertex.



The pressure difference (or potential difference) across AC is equal to:

(the pressure at C) – (the pressure at A) = $P_C - P_A = 70 - 50 = 20$.

Similarly,

the pressure difference across CD is $P_D - P_C = 90 - 70 = 20$;

the pressure difference across DA is $P_A - P_D = 50 - 90 = -40$.

Adding these numbers together, we see that the sum of the pressure differences around the circuit $ACDA$ is zero. In fact, the sum of the pressure differences around any circuit is zero — this is the analogue of Kirchhoff's voltage law for hydraulic networks. ■

In the next section we show how each component of a physical system — and hence the whole network — can be represented by an 'oriented' graph. With each edge of the graph we can associate a through variable and an across variable. These variables are related by the 'component equations'.

The component equations relate the through and across variables for a single component. For example, if the component is an electrical resistor, the appropriate component equation is **Ohm's law**:

$$\frac{\text{voltage}}{\text{current}} = \text{a constant (the resistance).}$$

For a spring, the corresponding component equation is **Hooke's law**:

$$\frac{\text{tension}}{\text{extension}} = \text{a constant (the stiffness).}$$

In this unit we consider only systems which obey Kirchhoff's laws or their analogues. We can use these laws to derive two further sets of equations, one set relating the through variables in the system (the *vertex law equations*) and the other relating the across variables (the *cycle law equations*).

Given these three sets of equations — the component equations, the vertex law equations, and the cycle law equations — we can set about solving them. That is, if we know the values of some of the edge variables of our graph (say, the voltages at particular points in an electrical network), then we can work out the values of all the other edge variables — the voltages and currents in this case. The aim of this unit is to describe how we tackle this type of problem.

Generally, we shall use electrical networks as examples to illustrate the solution techniques. You should, however, always bear in mind that the methods we describe are completely general, in that they can be used to work out the values of edge variables in any physical system that obeys Kirchhoff's laws. When we have worked out values for the edge variables, we may then interpret them as currents and voltages, flows and pressures, forces and velocities, and so on, depending on the type of system in hand.

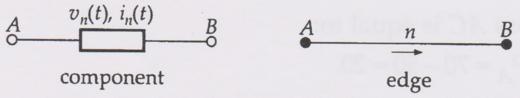
1.2 Graphical representation of components

~~Not Assessed~~

Graphical representation of 2-terminal components

In this section we show how 2-terminal components can be represented graphically. Later in this section we show that a component with more than two terminals (a **multi-terminal component**) can be represented graphically by treating it as a number of 2-terminal components.

A 2-terminal component can be represented by two vertices joined by a single edge. This is illustrated in the following diagram which shows a resistor and its representation as the edge of a graph:



Associated with the edge of the graph are two **edge variables** — one through variable and one across variable. In the case of the resistor shown above, the across variable is the voltage $v_n(t)$ across the resistor, and the through variable is the current $i_n(t)$ flowing through the resistor. We use the subscript n when we wish to indicate that the resistor is the nth component of a particular system. To avoid cluttered diagrams, we label the corresponding edge simply by the letter n.

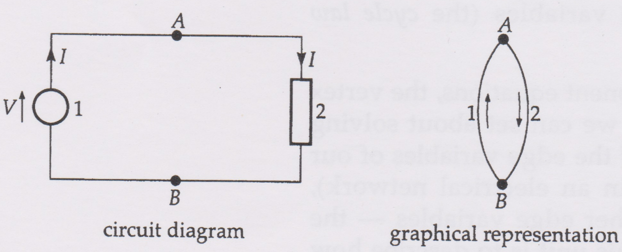
We have introduced the letter t to refer to time, since the currents and voltages are often time-dependent. However, we usually omit t and write just v_n and i_n.

We also need a *reference direction* for the edge variables. We get this by assigning an arbitrary direction to each edge, and we indicate it by drawing an arrow alongside the edge. Currents which actually flow through the component corresponding to the edge in the direction of the arrow are regarded as positive, and currents in the opposite direction are regarded as negative. Similarly, voltage differences between the terminals corresponding to two vertices are always taken as the voltage at the tail of the arrow minus the voltage at the head. Thus if the voltage at the tail is greater than the voltage at the head, the voltage difference is positive: if the voltage at the tail is less, then the voltage difference across the edge is negative. It is important to note that the reference direction is purely arbitrary and does not necessarily represent the actual direction of the current flow or voltage difference.

Note that our convention of using the reference arrow to specify the direction of a voltage is the opposite to that normally used in electrical circuit theory: an arrow on a *circuit* diagram indicates a positive voltage if the voltage at the head is greater than the voltage at the tail. Here we use a different convention because we are using a single reference arrow to indicate the direction of both current and voltage. This usage is illustrated in Example 1.2.

Example 1.2

Consider a system consisting of two 2-terminal components — a battery and a resistor. The following figure shows the circuit diagram and its graphical representation.



The circuit diagram shows the actual direction of flow of the current I and the actual direction of the voltage V across the battery. In the graph, the edge labelled 1 corresponds to the battery, and the edge labelled 2 corresponds to the resistor. Since both components are connected together at terminals A and B in the circuit diagram, the corresponding edges are connected to the same vertices in the graph.

The edge variables associated with edge 1 are the current i_1 flowing through the battery, and the voltage v_1 across the battery. Similarly, the variables associated with edge 2 are the current i_2 flowing through the resistor, and the voltage v_2 across the resistor.

Consider edge 1. The reference direction is the same as the direction of flow of the current through the battery, so we have

$$i_1 = I.$$

In the circuit diagram, the arrow by the battery points towards terminal A, indicating that A is the positive terminal of the battery. Therefore the potential at terminal A is V volts higher than the potential at terminal B. In the graph representation, the arrow associated with edge 1 points from B towards A. The voltage v_1 is equal to the voltage at the tail of the arrow (B) minus the voltage at the head (A) — that is,

$$v_1 = -V.$$

Note that, although the reference arrow on the graph points in the same direction as the arrow on the circuit diagram, the expression for v_1 has a minus sign — this is because the sign conventions are different in the two cases.

Now consider edge 2. The reference direction is again the same as the direction of the flow of the current through the resistor, so we have

$$i_2 = I.$$

Terminal A of the resistor is connected to the positive battery terminal and so is at a higher potential than terminal B. Thus the voltage at the tail of the reference arrow on the graph is greater than the voltage at the head, so

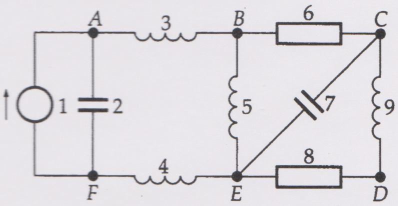
$$v_2 = V.$$

We have expressed all four of the edge variables in terms of the two quantities given on the original circuit diagram. ■

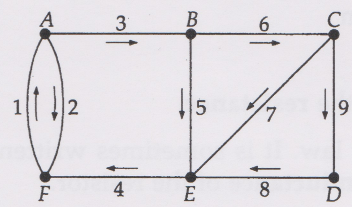
We have now given examples of the graphical representation of a single 2-terminal component, and of two such components connected together. We can extend this method of representation to more complicated networks. In fact, *any system consisting entirely of 2-terminal components can be represented by a graph whose edges correspond to the components and whose vertices correspond to the terminals.*

Example 1.3

The following electrical network contains nine components — an independent voltage source, two resistors, two capacitors and four inductors. Replacing each of these nine components by an edge, and assigning an arbitrary direction to each of these edges, we obtain the corresponding graph:



electrical network

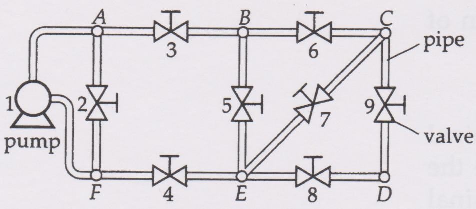


graphical representation

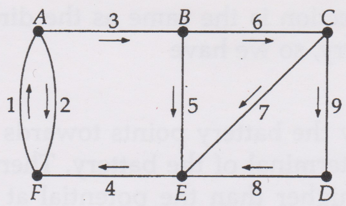
■

You may be wondering why we have not drawn the direction arrows on the edges (rather than beside them), referred to the edges as 'arcs', and called the diagram a 'digraph', rather than a graph. The reason is that for much of our discussion the directions are irrelevant: for example, we shall need to refer to cycles such as $BCEB$ or $BCDEB$, and these can be considered as cycles only when we ignore the directions of the arrows. A diagram of this kind, which is something between a graph and a digraph, is called an **oriented graph**.

Oriented graphs can also be used to represent systems other than electrical systems. For example, the above oriented graph also corresponds to the following hypothetical hydraulic power system, in which the terminals A, B, C, D, E and F are pipe junctions, and each segment of pipe contains a valve. The 'voltages' $v_1, \ldots, v_9$ are across variables representing the pressure drops across the valves, and the 'currents' $i_1, \ldots, i_9$ are through variables representing the flows through the corresponding valves.



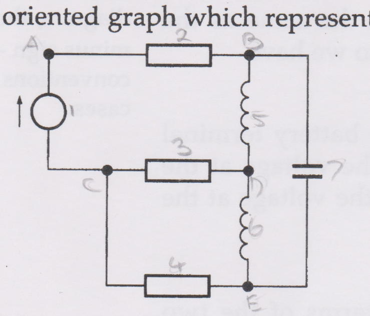
hydraulic power system



oriented graph

Problem 1.1

Label the components and the terminals of the following electrical network, and draw an oriented graph which represents it.



Component equations for 2-terminal components

We now turn to the equations relating the two edge variables for each component — the **component equations**. Suppose that we measure the voltage across a 2-terminal component such as a resistor for different values of current flowing through it. We can then try to model the relationship between the voltage and current by an equation. Such an equation is called a *component equation* since it is characteristic of the component itself, as opposed to Kirchhoff's laws, which are a general property of any electrical network.

We start by looking at four types of electrical component — *resistors, capacitors, inductors* and *independent sources.*

A **resistor** is modelled by a 2-terminal component whose edge variables v and i satisfy a relationship of the form

$$v = Ri,$$

where R is a constant, usually called the **resistance**.

This relationship is known as Ohm's law. It is sometimes written in the form $i = Gv$, where G $(= 1/R)$ is the **conductance** of the resistor.

A **capacitor** is modelled by a 2-terminal component whose edge variables v and i satisfy a relationship of the form

$$i = C\frac{dv}{dt},$$

where C is a constant, usually called the **capacitance**.

An **inductor** is modelled by a 2-terminal component whose edge variables v and i satisfy a relationship of the form

$$v = L\frac{di}{dt},$$

where L is a constant, usually called the **inductance**.

Note that the component equations for a resistor, a capacitor and an inductor are all *linear equations*. Note also that R, C and L are *constants* — we are assuming that the behaviour of the components is *time-invariant*. The concepts of *linearity* and *time-invariance* are discussed briefly at the end of this section.

To model components such as batteries and electric generators, we use the concept of an **ideal independent source.** This is an idealized 2-terminal component which keeps the value of one of the edge variables at a prescribed level independent of the value of the other edge variable. An ideal independent *voltage* source, for example, maintains a *prescribed voltage* across its terminals, irrespective of the size of the current flowing between them. Under certain conditions, a battery can be modelled in terms of such a source. Similarly, an ideal independent *current* source maintains a *prescribed current* irrespective of the value of the voltage across its terminals.

The prescribed level of an independent source may not necessarily be constant; for example, for an independent voltage source, the prescribed voltage may be a sinusoidal voltage of the form $v_0 \sin \omega t$, where v_0 and ω are constants.

Some 2-terminal components of other types of system such as mechanical or hydraulic systems can be considered as analogues of these electrical components. The electrical analogues of two components of a mechanical system are considered in the following example and problem.

Example 1.4

Suppose that a force f accelerates a body of mass m. Then, according to Newton's second law of motion, the acceleration a is given by the equation

$$f = ma$$

or

$$f = m\frac{du}{dt},$$

where u is the velocity of the mass. In this case, f is the through variable and u is the across variable. This equation is of the same form as the equation

$$i = C\frac{dv}{dt}$$

for a capacitor. We may therefore consider the force f to be analogous to the current i (also a *through variable*), the velocity u to be analogous to the voltage v (also an *across variable*), and the mass m to be analogous to the capacitance C. The following table shows the analogous quantities for the two components.

	mechanical	electrical
component	mass	capacitor
through variable	f	i
across variable	u	v
component equation	$f = m\frac{du}{dt}$	$i = C\frac{dv}{dt}$

■

Problem 1.2

For a spring which obeys Hooke's law, the extension x of the spring produced by a force f is given by the equation

$$f = kx,$$

where k is a constant (the stiffness of the spring). If the rate of change of the extension of the spring (that is, the velocity $u = \frac{dx}{dt}$) is considered to be analogous to a voltage, which electrical component is analogous to the spring? Construct a table of analogous quantities similar to the above table.

For the remainder of this unit, all the examples we consider are electrical. We can obtain corresponding results for other physical systems by considering the appropriate electrical analogues.

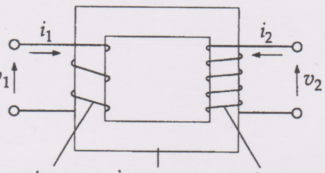
Multi-terminal components

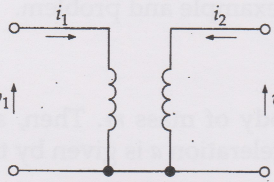
A **multi-terminal component** is a component which has more than two terminals, and which cannot therefore be represented by a single edge joining two vertices. Before showing how such a component can be represented graphically, we consider two important types of electrical multi-terminal component — *transformers* and *transistors*.

It is not necessary to be familiar with the technical details given below.

Transformers

A **transformer** is a device used extensively for changing the voltage at which electrical energy is supplied. A schematic diagram of a transformer consisting of two coils wound on an iron core is shown below.





ideal transformer

Under certain circumstances a transformer can be modelled by an *ideal transformer* which satisfies 'ideal' conditions such as having zero-resistance windings and no magnetic field losses.

A circuit diagram of an ideal transformer is shown above. Notice that one of the input terminals is connected to one of the output terminals; a transformer connected in this way can be regarded as a 3-terminal component. The equations relating the voltages and currents shown on the diagram, as derived from electromagnetic theory, are

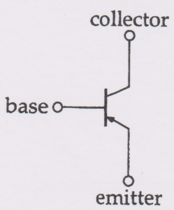
$$\frac{v_1}{v_2}=\frac{n_1}{n_2} \quad \text{and} \quad \frac{i_1}{i_2}=-\frac{n_2}{n_1},$$

Note that v_1, i_1, v_2 and i_2 are usually functions of time.

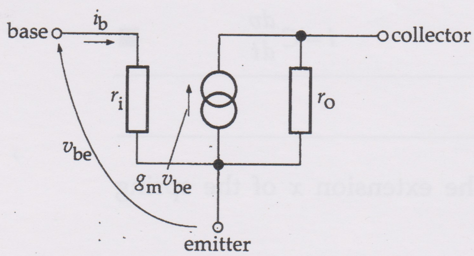
where n_1 and n_2 are the numbers of turns in the two coils.

Transistors

A **transistor** has three external connections called the *emitter*, the *base* and the *collector*. Under certain circumstances, its behaviour can be modelled by the circuit shown in the following diagram.



(a) pnp transistor



(b) controlled source representation. g_m is the transfer conductance or 'mutual conductance' of the transistor

Diagram (a) shows the symbol commonly used to represent a pnp type of transistor. Diagram (b) shows the controlled source representation. When a small voltage v_{be} is applied between the emitter and the base, under certain circumstances a greatly amplified voltage will appear between the emitter and the collector; in other words, the transistor acts as an

A *controlled source* is an independent source whose output level is not constant but is controlled by the value of a particular voltage or current.

amplifier. This is represented in diagram (b) by a current source which is controlled by the value of the input voltage v_{be}. The current flowing in the source is equal to $g_m v_{be}$, where g_m is a constant for particular operating conditions of the transistor. Notice that resistors have been added to the controlled source representation to model the input resistance r_i and the output resistance r_o of the transistor. We can use this representation to obtain relations between the voltages and currents. Since the input circuit consists of just a resistance r_i, the input current i_b is given by

$$i_b = \frac{v_{be}}{r_i}.$$

The output current i_c is equal to the source current $g_m v_{be}$ minus the current flowing through the resistor r_o, that is,

$$i_c = g_m v_{be} - \frac{v_{ce}}{r_o},$$

where v_{ce} is the output voltage between the collector and the emitter.

Graphical representation of multi-terminal components

We have seen how a 2-terminal component can be represented graphically by a single edge. We now show that a multi-terminal component can be regarded as a number of 2-terminal components, and can therefore be represented by a graph with a number of edges. To do this, we need to make use of Kirchhoff's current and voltage laws which we stated earlier, and repeat here for convenience.

Kirchhoff's current law

The current flow into any vertex of an electrical network is equal to the current flow out of it.

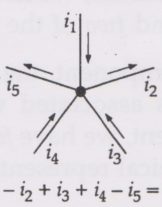
Kirchhoff's voltage law

The algebraic sum of the potential differences across all the components around any circuit (cycle) in an electrical network is zero.

These two laws can be formulated in graph-theoretical terms as follows.

Vertex law

The algebraic sum of the through variables associated with the edges at each vertex of an oriented graph is zero.

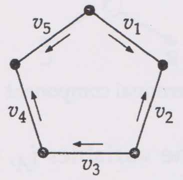


$i_1 - i_2 + i_3 + i_4 - i_5 = 0$

Here 'the algebraic sum' means that the orientation of the edges must be taken into account, so if edges oriented *into* a vertex are counted as *positive*, then edges oriented *out of* a vertex are counted as *negative*.

Cycle law

The algebraic sum of the across variables associated with the edges in any cycle of an oriented graph is zero.

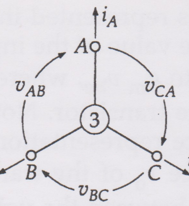


$v_1 - v_2 + v_3 + v_4 - v_5 = 0$

Here 'the algebraic sum' means that we choose a particular direction around a cycle (say clockwise) and count edges oriented in the *same direction* as *positive* and edges oriented in the *opposite direction* as *negative*.

We know that a 2-terminal component can be represented by a single edge. Associated with the edge are two variables: the voltage across the terminals of the component (across variable) and the current flowing

through the component (through variable). We now consider the variables associated with a 3-terminal component.

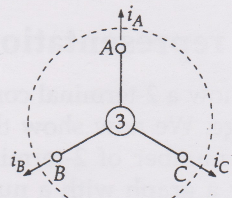


We can measure the current flowing out of each of the three terminals A, B and C; the currents are shown on the diagram as i_A, i_B and i_C. Similarly, we can measure the voltage across each pair of terminals; the voltages are shown as v_{AB} for terminals A and B, v_{BC} for terminals B and C, and v_{CA} for terminals C and A.

At first sight it would seem that we need six variables to characterize a 3-terminal component — three voltages and three currents. However, these six variables are not all independent since we can obtain relationships between them by applying Kirchhoff's two laws, as follows.

Kirchhoff's current law states that the total current flowing into a vertex is equal to the total current flowing out of the vertex. It can be deduced that the total current flowing into the region bounded by broken lines in the figure below is equal to the total current flowing out of that region.

We show in Section 2 that Kirchhoff's current law can be applied to any cutset in an oriented graph.



Applying Kirchhoff's current law in this way to the currents flowing across the boundary in the figure, we obtain the equation

$$i_A + i_B + i_C = 0$$

or

$$i_A = -i_B - i_C.$$

A consequence of this last equation is that if we measure i_B and i_C then we do not need to measure i_A; we can calculate it from the equation. So only *two* of the three current variables are independent.

We can apply Kirchhoff's voltage law to the cycle $ABCA$. We thus obtain the equation

We can assume that the component is connected to other components or to measuring instruments, so that there is such a cycle.

$$v_{AB} + v_{BC} + v_{CA} = 0$$

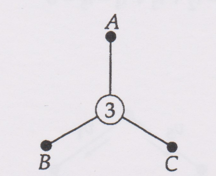
or

$$v_{BC} = -v_{AB} - v_{CA}.$$

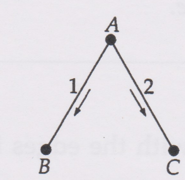
Thus if we measure v_{AB} and v_{CA}, then we do not need to measure v_{BC}; we can calculate it from the equation. So only *two* of the three voltage variables are independent.

Thus, of the six current and voltage variables, only four are independent — two current variables and two voltage variables. It follows that, to describe completely the behaviour of the 3-terminal component as measured at its terminals, we need to specify the values of only *two* of the current variables and *two* of the voltage variables.

For a 2-terminal component, the graphical representation is a single edge with two variables associated with it (a current and a voltage). For a 3-terminal component, we have *four* independent variables, so we need *two* edges for the graphical representation. One possibility is shown below:



3-terminal component



graphical representation

We can express the variables i_A, v_{AB}, etc. in terms of the edge variables v_1, i_1 and v_2, i_2 as follows.

In the graphical representation, the arrow on edge 1 points towards the vertex B. This means that the current i_1 is directed towards vertex B and therefore flows *out* of the 3-terminal component via terminal B — it is therefore equal to the current i_B.

Clearly, the current flowing out of terminal B is equal to i_1, so

$$i_B = i_1.$$

Similarly,

$$i_C = i_2,$$

and, since $i_A = -i_B - i_C$, we have

$$i_A = -i_1 - i_2.$$

The voltage across terminals A and B is clearly equal to v_1, so

$$v_{AB} = v_1.$$

Similarly,

$$v_{CA} = -v_2,$$

and, since $v_{BC} = -v_{AB} - v_{CA}$, we have

$$v_{BC} = v_2 - v_1.$$

We showed earlier that a 2-terminal component is represented by *one* edge and has *one* component equation. We have just shown that a 3-terminal component is represented by *two* edges and, by analogy with the 2-terminal component, has *two* component equations. As a result, we can regard any 3-terminal component as being composed of two 2-terminal components, one corresponding to each edge of the graphical representation. To represent the 3-terminal component shown diagrammatically above, we chose the two edges AB and AC. Instead, we could have chosen the two edges AB and BC, or the two edges BC and AC. In the following problem we ask you to find relationships between the variables for the different choices of pairs of edges.

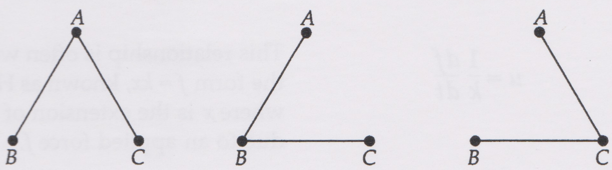
Problem 1.3

The following graphs represent the same 3-terminal component:

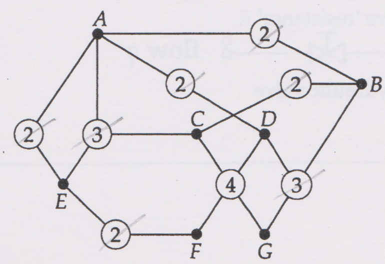
(a) (b)

In each case, find i_a, i_b, v_a and v_b in terms of the variables i_1, i_2, v_1 and v_2 defined above.

The method of representation just described can be extended to the graphical representation of n-terminal components for larger values of n. Just as a 3-terminal component can be represented by a tree with two edges, so an n-terminal component can be represented by a tree with $n-1$ edges. Moreover, we have seen that each of the labelled trees with three vertices (shown below) is a suitable representation of the 3-terminal component.



Similarly, any labelled tree with n vertices is a suitable representation of an n-terminal component. Thus *any n-terminal component can be regarded as n–1 2-terminal components*, one corresponding to each edge of the tree.



Problem 1.4

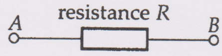
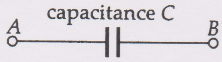
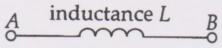
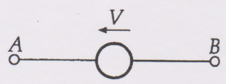
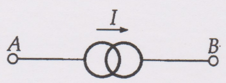
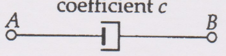
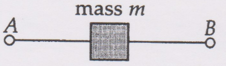
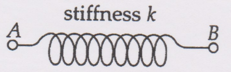
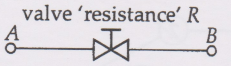
(a) Find a graphical representation of the system shown in the margin.

(b) How many component equations are associated with this system?

5+4+3

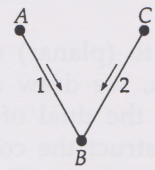
We conclude this subsection by listing some important types of component, together with their graphical representations and their corresponding component equations.

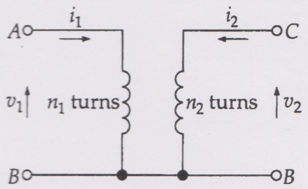
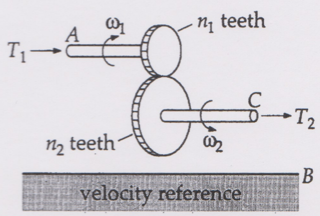
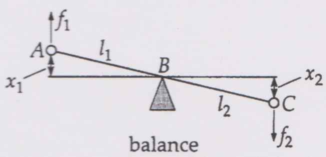
Table 1.2

two-terminal component	graphical representation $A \longrightarrow B$: through variable	across variable	component equation
resistance R  resistor	current i	voltage v	$v = Ri$
capacitance C  capacitor	current i	voltage v	$i = C\dfrac{dv}{dt}$
inductance L  inductor	current i	voltage v	$v = L\dfrac{di}{dt}$
 voltage source	current i	voltage v	$v = V$
 current source	current i	voltage v	$i = I$
damping coefficient c  damper	force f	velocity u	$f = cu$
mass m  mass	force f	velocity u	$f = m\dfrac{du}{dt}$
stiffness k  spring	force f	velocity u	$u = \dfrac{1}{k}\dfrac{df}{dt}$
valve 'resistance' R  hydraulic valve	flow q	pressure p	$p = Rq$

This relationship is often written in the form $f = kx$, known as Hooke's law, where x is the extension of the spring due to an applied force f.

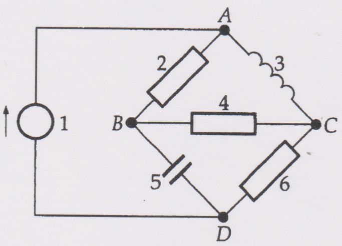
graphical representation



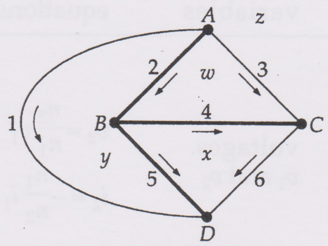
three-terminal component	through variables	across variables	component equations
 ideal transformer	currents i_1 and i_2	voltages v_1 and v_2	$v_2 = \frac{n_2}{n_1} v_1$ $i_2 = -\frac{n_1}{n_2} i_1$
 gear train (friction and inertia neglected)	torques T_1 and T_2	angular velocities ω_1 and ω_2	$T_2 = \frac{n_2}{n_1} T_1$ $\omega_2 = -\frac{n_1}{n_2} \omega_1$
 balance	forces f_1 and f_2	displacements x_1 and x_2	$f_2 = \frac{l_1}{l_2} f_1$ $x_2 = -\frac{l_2}{l_1} x_1$

1.3 Dual electrical networks

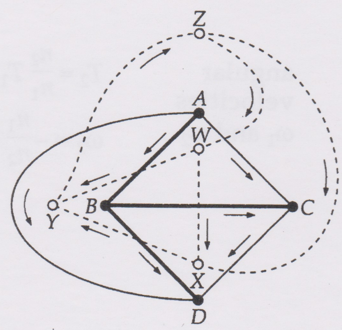
The idea of *duality* can be applied to (planar) electrical networks as follows. Given an electrical network, we draw a plane drawing of a corresponding oriented graph, form the dual of this planar graph as described in *Graphs 3*, and then construct the corresponding network, which we call the **dual network**. The method is illustrated below.

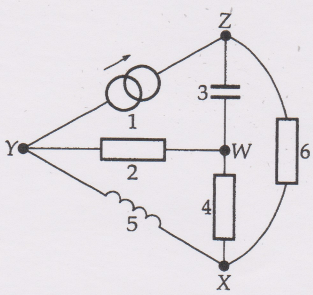


given network

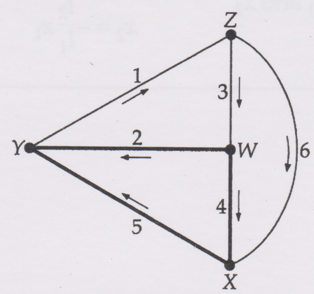


oriented graph G



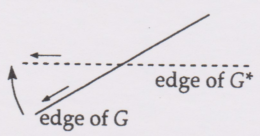


dual network



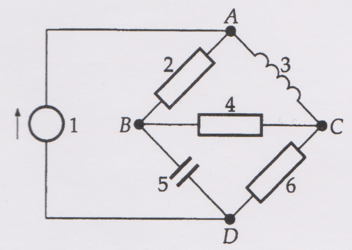
dual oriented graph G^*

We start with a given network N and draw the oriented graph; the oriented graph G has four faces, which we have labelled w, x, y and z. We construct the dual oriented graph G^* by drawing four vertices W, X, Y and Z corresponding to the four faces, and drawing an edge joining each pair of vertices separated by an edge in G. However, we are now dealing with *oriented* graphs, so we need to associate an arrow with each edge in G^*. The rule for doing this is illustrated below.

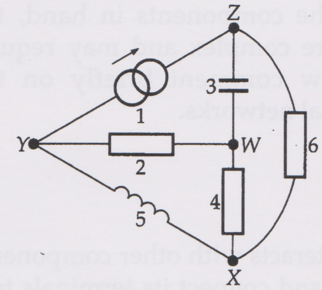


go *clockwise* from edge of G
to edge of G^* in *same* direction

Finally, we construct the dual network N^* from the dual oriented graph. It turns out that *voltage* in the original network N corresponds to *current* in the dual network N^* and *vice versa*. Since these roles are interchanged, the roles of v and i in the component equations are interchanged, so we obtain the following correspondences between the variables and components in the two networks N and N^*.



network



dual network

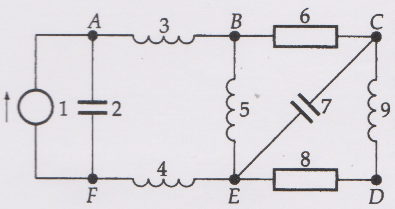
network N	dual network N^*
voltage	current
current	voltage
resistor (resistance R)	resistor (resistance $1/R$)
capacitor (capacitance C)	inductor (inductance C)
inductor (inductance L)	capacitor (capacitance L)

In Section 2 we find further correspondences between the fundamental equations for voltages and currents in the two networks.

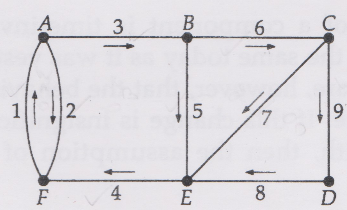
Problem 1.5

Construct the dual of each of the following networks:

(a)

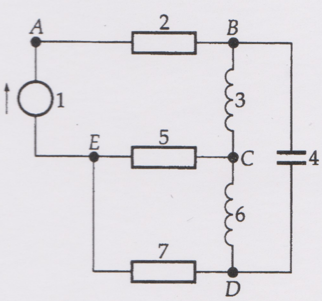


electrical network

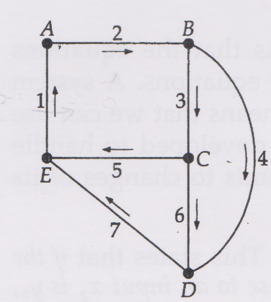


oriented graph

(b)



electrical network



oriented graph

1.4 Assumptions in modelling component behaviour

In our discussion of component equations, we made a number of implicit assumptions about component behaviour. In particular, we assumed that the behaviour of a component does not change if it is moved from one system to another, and that the behaviour is time-invariant and linear. If these assumptions are not applicable to the components in hand, the mathematics will often turn out to be more complex and may require special manipulative techniques. We now comment briefly on the assumptions made in our treatment of physical networks.

System independent behaviour

One way of finding out how a component interacts with other components in a system is to remove it from the system and connect its terminals to a test system which can be used to take measurements. In this way we can construct a model of its behaviour, expressed as the component equation. There is, however, a very important implied assumption in doing this. We are assuming that the component will behave in the same way in our test system as in the original system. Otherwise, we would not be able to use the information obtained from the test to work out how the component behaves in the original system. Another way of putting this is to say that we must be careful that the model (or component equation) which we use is appropriate for the system in question. For example, one system may operate in a different type of environment from a second system, with a different ambient temperature, humidity or air pressure, and this may significantly affect the component behaviour. Even if the environments of two systems are the same, we may not be able to use the same model of a component for both systems. For example, a model of a transistor which is suitable for describing its behaviour in a hi-fi amplifier may not be suitable for describing its behaviour in a computer.

Time-invariance

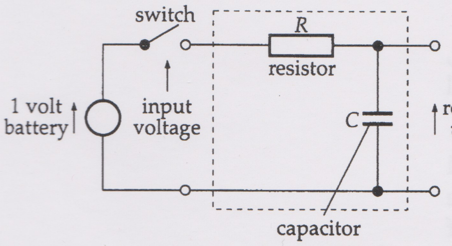
The assumption that the behaviour of a component is time-invariant means that we assume its behaviour is the same today as it was yesterday and will be tomorrow. We must be aware, however, that the behaviour of real components will change with time. If this change is insignificant in the application we are concerned with, then the assumption of time-invariance is reasonable.

Linearity

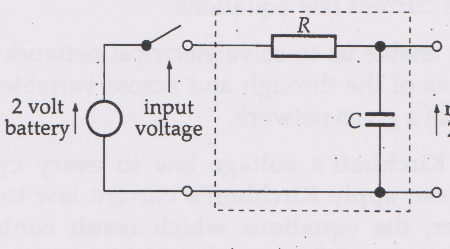
Mathematically, the assumption of linearity means that the equations describing the behaviour of a component are *linear* equations. A system made up of linear components is itself linear. This means that we can use the many mathematical techniques that have been developed to handle linear equations to work out how the system responds to changes of its inputs.

A linear system obeys the *principle of superposition*. This states that *if the response of a system to an input x_1 is y_1, and the response to an input x_2 is y_2, then the response of the system when both inputs are applied together is $y_1 + y_2$*. Another way of looking at this is to consider the response of a system to a particular input. If the system is linear, and if, for example, the size of the input is doubled, then the resulting response will also be doubled.

The following diagrams illustrate this principle for a simple electrical system in which the behaviour of the components can, for our purposes, be modelled by linear equations. In each case, a capacitor and a resistor are connected via a switch to a battery.

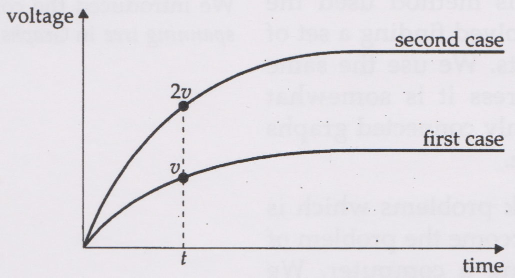


(a) first case



(b) second case

In the first case, the voltage of the battery is 1 volt, and the system's response — the voltage v measured across the output terminals — on closing the switch is shown below. If we now replace the battery by one of 2 volts and repeat the experiment, then the output voltage at any instant will be double what it was in the first case.



After studying this section, you should be able to:

- understand what is meant by *through* and *across variables* and identify them in physical systems;
- give examples of 2-terminal components and state the corresponding *component equations;*
- give examples of multi-terminal components and explain why a multi-terminal component can be regarded as a number of connected 2-terminal components;
- explain how physical systems can be represented by oriented graphs, and draw the oriented graph corresponding to a given system;
- construct the *dual network* of a given electrical network.

2 Electrical networks: matrix equations

In the previous section we showed that an electrical network consisting of a number of interconnected components can be represented by an oriented graph. We also discussed the equations relating the voltages and currents associated with each component — the component equations.

Remember that voltages are across variables and currents are through variables.

In this section we discuss two further sets of equations — the equations obtained by applying Kirchhoff's voltage law, and the equations

obtained by applying Kirchhoff's current law to the oriented graph. In Section 3 we show that these three sets of equations:

- the component equations,
- Kirchhoff's voltage law equations,
- Kirchhoff's current law equations,

are sufficient to enable us to solve electrical network problems — that is, to find the values of the through and across variables associated with all the components of a given network.

We can apply Kirchhoff's voltage law to every cycle of the oriented graph, and we can apply Kirchhoff's current law to every vertex of the graph. However, the equations which result contain a good deal of redundant information. What we should like is a systematic method for obtaining just enough equations to enable us to solve electrical network problems — or, to put it another way, to obtain sets of equations which contain all the information which Kirchhoff's laws can tell us about the network, without any redundancy.

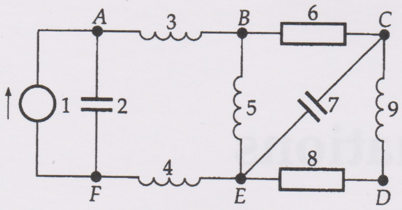
Such a systematic method was given by Kirchhoff in 1847 in a paper entitled 'On the solution of the equations obtained from the investigation of the linear distribution of galvanic currents'. His method used the concept of what is now called a *spanning tree* and involved finding a set of fundamental cycles and a set of fundamental cutsets. We use the same method, although the language in which we express it is somewhat different from that used by Kirchhoff. We consider only connected graphs — this means that we can always find a spanning tree.

We introduced the concept of a *spanning tree* in *Graphs 2*.

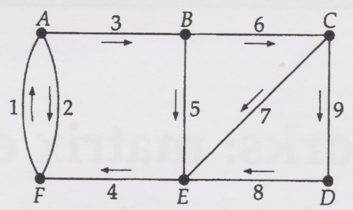
Our aim is to describe a method of solving network problems which is appropriate for computer analysis, so we need to overcome the problem of obtaining the fundamental cycles and cutsets using a computer. We describe a method which enables us to obtain the fundamental cycles and cutsets directly from the incidence matrix of the oriented graph. The great advantage of this method is that the only information about the oriented graph which we have to feed into the computer is the incidence matrix itself, and we can obtain the incidence matrix by inspection from the graph.

2.1 Kirchhoff's voltage law equations

We can apply Kirchhoff's voltage law to any cycle of the oriented graph of an electrical network. But if we apply this law to all the cycles, we end up with a set of equations which contain a lot of redundant information. We demonstrate this for the following electrical network which we use as an illustrative example throughout this section.



electrical network



oriented graph

Let us start by applying the voltage law to the cycles *BCEB*, *CDEC* and *BCDEB* of the oriented graph.

Recall that in going around each cycle we take a voltage to be positive if it is oriented in the same direction as the cycle, and negative if it is not.

This gives the following voltage equations.

BCEB	$v_6 + v_7 - v_5 = 0$
CDEC	$v_9 + v_8 - v_7 = 0$
BCDEB	$v_6 + v_9 + v_8 - v_5 = 0$

The last of these equations is simply the sum of the other two, and so gives no further information about the voltages.

Similarly, applying the voltage law to the cycles *ABEFA*, *BCEB*, *CDEC* and *ABCDEFA* (using the right-hand edge *AF*), we obtain the following voltage equations.

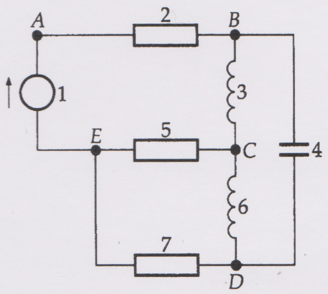
ABEFA	$v_3 + v_5 + v_4 - v_2 = 0$
BCEB	$v_6 + v_7 - v_5 = 0$
CDEC	$v_9 + v_8 - v_7 = 0$
ABCDEFA	$v_3 + v_6 + v_9 + v_8 + v_4 - v_2 = 0$

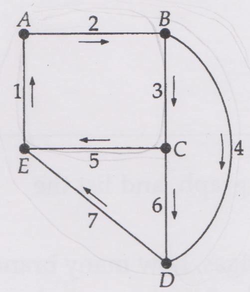
Again, the last of these equations is the sum of the others, and so gives no further information about the voltages.

In each of these cases, we can write each equation as the sum or difference of the other equations, and we express this by saying that the equations are *linearly dependent*. More generally, a number of equations are said to be linearly dependent if at least one of them 'depends' on the others, in the sense that it can be obtained by adding or subtracting multiples of the others. If none of the equations depends on the others in this way, then the equations are said to be *linearly independent*. For example, the voltage equations for the cycles *ABEFA*, *BCEB* and *CDEC* are linearly independent, since each contains a term contained in neither of the other two equations.

Problem 2.1

Consider the following oriented graph, which corresponds to the network in Problem 1.1.


electrical network


oriented graph

(a) Write down the voltage equation for each of the seven cycles of the graph.

(b) What is the maximum number of these seven equations which are linearly independent?

2.2 Fundamental cycles

We now investigate the problem of finding a set containing the maximum number of Kirchhoff's voltage law equations which are linearly independent. What we need to do is to find a set of cycles in an oriented graph which give rise to such a set of linearly independent equations. We say that a number of cycles in a graph are *linearly independent* if the corresponding voltage equations are linearly independent. For instance, in our illustrative example, the cycles *ABEFA*, *BCEB* and *CDEC* are linearly independent, since they correspond to linearly independent equations. It follows that, in order to find a set of linearly independent equations for the network, it is enough to identify the corresponding linearly independent cycles. We can find such a set of cycles quite easily; to do so, we use Kirchhoff's idea of a spanning tree.

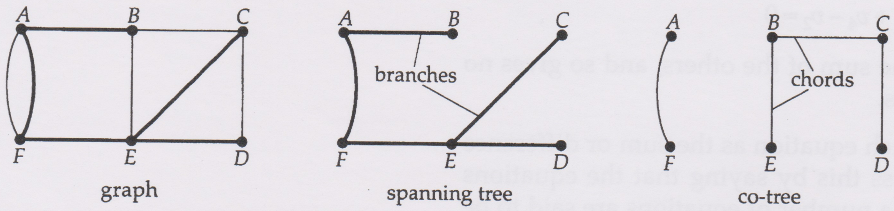
Definitions

Let G be a connected graph. A **spanning tree** in G is a subgraph of G which includes all the vertices of G and is also a tree. The edges of the tree are called **branches**. If we remove from G all the edges of a spanning tree, the remaining subgraph is called a **co-tree** and its edges are called **chords**.

The number of spanning trees in a graph can be very large; for instance, the graph in our example (shown below) has exactly 50 different spanning trees.

Note that a co-tree is, in general, not a tree.

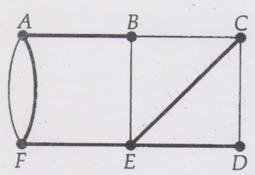
The following diagram shows the graph in our example, one of its spanning trees, and the associated co-tree.



For any connected graph G, we can find a spanning tree systematically. We start by choosing any cycle in G and removing one of its edges.

If there are no cycles, the graph G is itself a spanning tree.

Since we cannot disconnect a graph by removing just one edge from a cycle, we still have a connected graph. We now repeat the process until there are no cycles left, and there is our spanning tree. For the above graph, removal of the edge CD from the cycle $CDEC$, the edge BC from the cycle $BCEB$, the edge BE from the cycle $ABEFA$, and one of the edges FA from the cycle AFA, leaves the spanning tree shown.



Problem 2.2

(a) Find two more spanning trees in the above graph, and list the branches and chords in each case.

(b) If a connected graph has n vertices and m edges, how many branches and chords are there in each spanning tree? n-1 edges branches chords m-(n-1)

(c) What happens if we add a chord to a spanning tree? Illustrate your answer using the spanning tree given in our example.

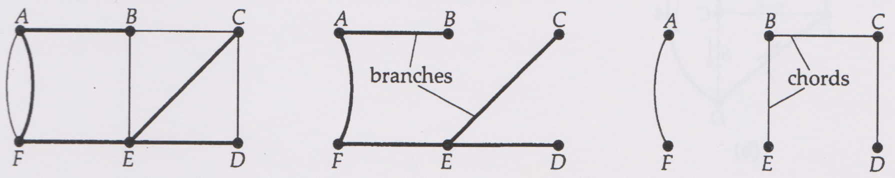
The last part of Problem 2.2 shows us how to find a set of linearly independent cycles. We first choose a spanning tree; then we add one of the chords to the tree — this produces exactly one cycle. We repeat this procedure for each chord. If the graph has k chords of the spanning tree, then this procedure produces k cycles. Since each cycle contains one chord which is not contained in any of the other cycles, the resulting set of cycles must be linearly independent.

If we were seeking a set of equations to solve by hand, we would choose the spanning tree with some care. In particular, we would choose the tree so that in general each edge for which the across variable was known would be a branch of the tree, and each edge for which the through variable was known would be a chord. However, this selection is not necessary in the mathematical sense; it serves only to make the resulting equations quicker to solve by hand. We shall describe a method appropriate for solving physical network problems on a computer, where such considerations are unimportant, and so we do not need to follow a selection procedure: we simply choose any convenient spanning tree.

Since the set of cycles we have found using our spanning tree are linearly independent, the corresponding voltage law equations must also be linearly independent. This leads to the following definitions.

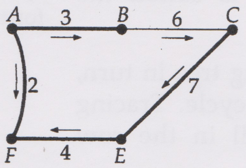
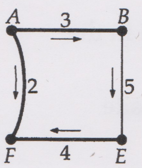
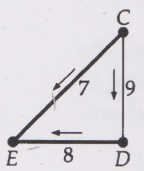
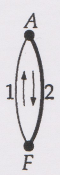
> ## Definitions
>
> Let G be a connected graph, and let T be a spanning tree of G. The set of **fundamental cycles** associated with T consists of the cycles of G each of which is obtained by adding a single chord to T. The corresponding voltage law equations are called the **fundamental cycle equations.**

For example, for the spanning tree shown below



graph   spanning tree   co-tree

we have the following table:

chord	fundamental cycle	diagram	fundamental cycle equation
BC	$ABCEFA$		$v_3 + v_6 + v_7 + v_4 - v_2 = 0$
BE	$ABEFA$		$v_3 + v_5 + v_4 - v_2 = 0$
CD	$CDEC$		$v_9 + v_8 - v_7 = 0$
FA	AFA		$v_2 + v_1 = 0$

Note that the cycles are taken in the same direction as the chords they contain. Thus we have written $ABEFA$ and $CDEC$ rather than $AFEBA$ and $CEDC$.

It follows from Problem 2.2(b) that, if G has n vertices and m edges, then any spanning tree gives rise to $m - n + 1$ chords, and so there are $m - n + 1$ cycles in a fundamental set. This number is the largest number of linearly independent cycles in G, and is called the **cycle rank** of G. It follows that there are $m - n + 1$ fundamental cycle equations, and that this is the largest possible number of linearly independent cycle equations. We conclude that, in any electrical network problem, we can obtain all the information contained in Kirchhoff's voltage law by:

(a) choosing a spanning tree;

(b) finding the fundamental cycles associated with it;

(c) finding the corresponding fundamental cycle equations.

An analogous result holds for the information contained in the generalized form of Kirchhoff's voltage law for other types of network.

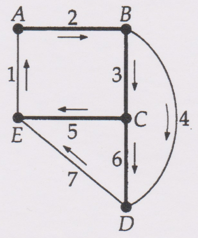
Problem 2.3

For each of the spanning trees shown below, write down:

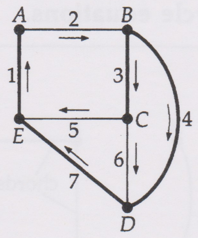
- a list of the chords;
- the corresponding fundamental cycles;
- the corresponding fundamental cycle equations.

In each case, show how to obtain the voltage law equation for the cycle *ABCDEA* from the fundamental cycle equations.

a) AE
BD
DE



(a)



(b)

Before leaving fundamental cycles, we show how the fundamental cycle equations can be represented in matrix form. The rows of the matrix correspond to the chords associated with a spanning tree, and the columns correspond to all the edges of the graph G. If G has n vertices and m edges, we get an $(m - n + 1) \times m$ matrix. This matrix is called the **fundamental cycle matrix**, and is denoted by $\mathbf{C}_f$.

It is usual to write the columns corresponding to the branches of the spanning tree first, followed by those for the chords.

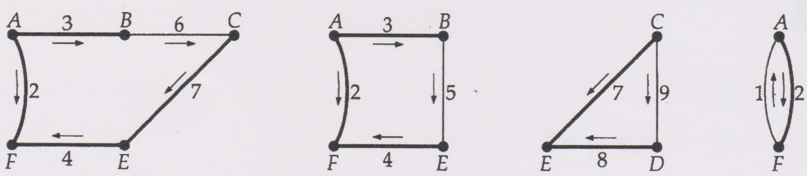
To construct this matrix, we add each chord to the spanning tree in turn, and look at the edges in the corresponding fundamental cycle. Tracing around this cycle in the direction of the chord, we fill in the row corresponding to the given chord as follows: we write

1 in each column corresponding to an edge of the cycle oriented in the same direction as the chord;

–1 in each column corresponding to an edge oriented in the opposite direction;

0 in each column corresponding to an edge which is not in the cycle.

For example, for the fundamental cycle *ABCEFA* illustrated below, the row corresponding to the chord *BC* has 1 in the columns corresponding to *AB*, *CE*, *EF* and *BC*, –1 in the column corresponding to *AF*, and 0 elsewhere.



Repeating this procedure for all four fundamental cycles, we obtain the following matrix:

$$
\begin{array}{c}
\\ BC \\ BE \\ CD \\ FA
\end{array}
\begin{array}{c}
\begin{array}{ccccccccc} AB & AF & CE & DE & EF & BC & BE & CD & FA \end{array} \\
\left[\begin{array}{ccccc|cccc}
1 & -1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
\end{array}
\left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} \text{chords}
$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{\text{branches}} \quad \underbrace{\qquad\qquad\qquad}_{\text{chords}}$$

Having found the fundamental cycle matrix $\mathbf{C}_f$, we can write the fundamental cycle equations in matrix form as

$$\mathbf{C}_f\,\mathbf{v} = \mathbf{0},$$

where $\mathbf{v}$ is the column vector of edge voltages, written in the same order as the columns of the matrix, and $\mathbf{0}$ is the appropriate zero vector. For example, the above matrix gives

$$\begin{array}{l}
\text{edge number} \rightarrow \quad 3 \quad 2 \quad 7 \quad 8 \quad 4 \quad 6 \quad 5 \quad 9 \quad 1 \\
\begin{bmatrix} 1 & -1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_3 \\ v_2 \\ v_7 \\ v_8 \\ v_4 \\ v_6 \\ v_5 \\ v_9 \\ v_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\end{array}$$

Multiplying the two matrices on the left, we obtain the following equation, which is equivalent to the four equations on page 27.

$$\begin{bmatrix} v_3 - v_2 + v_7 + v_4 + v_6 \\ v_3 - v_2 + v_4 + v_5 \\ -v_7 + v_8 + v_9 \\ v_2 + v_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

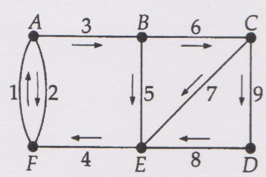
Problem 2.4

For each of the spanning trees in Problem 2.3, find the corresponding fundamental cycle matrix, and use it to derive the fundamental cycle equations in each case.

2.3 Kirchhoff's current law equations

We now turn our attention to Kirchhoff's current law, which states that the algebraic sum of the currents at each vertex of the network is zero. As with the voltage law, we find that if we apply the current law to every vertex of a graph, then we end up with more equations than we need.

For example, consider our oriented graph:



Applying the current law at each vertex, we obtain the following equations.

vertex A	$i_1 - i_2 - i_3 = 0$
vertex B	$i_3 - i_5 - i_6 = 0$
vertex C	$i_6 - i_7 - i_9 = 0$
vertex D	$i_9 - i_8 = 0$
vertex E	$i_5 + i_7 + i_8 - i_4 = 0$
vertex F	$i_2 + i_4 - i_1 = 0$

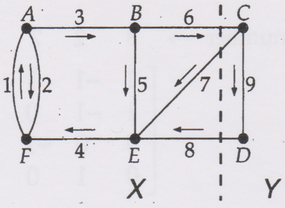
These six equations are not linearly independent: if we add any five of them, we get the sixth. However, any five of these equations are linearly independent.

We show later that for a graph with n vertices there are always exactly $n-1$ linearly independent equations.

In fact, for the solution procedures which we use here, it is more appropriate to apply Kirchhoff's current law to the cutsets of the graph, rather than to the vertices.

We defined a *cutset* of a connected graph in *Networks 1*.

To see what we mean by this, consider the cutset consisting of the edges BC, CE and DE of our graph:



Removal of the edges of this cutset disconnects the graph into two parts, X and Y. If we add the equations for the vertices in just one of these parts (Y, say), we find that the currents along those edges lying entirely within that part cancel out, and we are left with an equation relating the currents in the edges of the cutset.

For example, if we add together the above equations for the vertices C and D, the current i_9 cancels out, leaving the equation

$$i_6 - i_7 - i_8 = 0.$$

This equation relates the currents in the cutset edges BC, CE and DE. Note that we get the same equation if we add the equations for the vertices A, B, E and F in the set X.

We can generalize this idea, and use the current law to find an equation relating the currents in any cutset. If the cutset separates the graph into two parts X and Y, then the equation has the form

(the sum of the currents from X to Y)

− (the sum of the currents from Y to X)

= 0.

Or simply,

the algebraic sum of the currents in any cutset is zero.

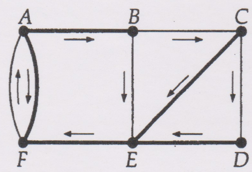
The word 'algebraic' means that we take account of the orientation of each edge.

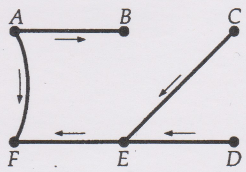
However, if we repeat this procedure for all the cutsets in the graph, we find that the resulting equations are not all linearly independent.

2.4 Fundamental cutsets

As with the cycle equations, we wish to find the largest possible number of linearly independent cutset equations. To do this, we find a set of fundamental cutsets. As with fundamental cycles, these are constructed using a spanning tree.

We illustrate the method with the graph and spanning tree we used previously:



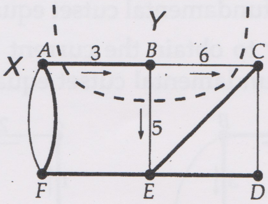
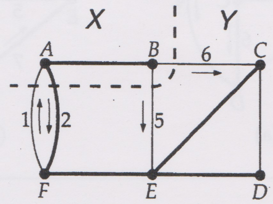
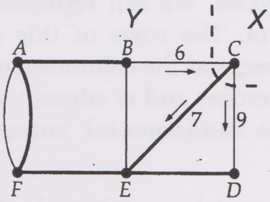
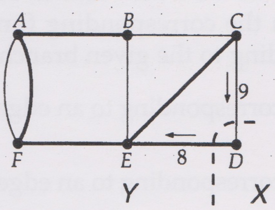
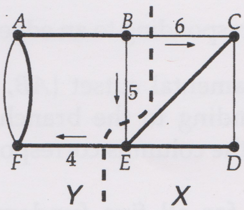


The removal of any branch separates the tree into two parts, X and Y; for example, the removal of the branch EF separates the tree into two parts, one containing the vertices C, D and E, and the other containing the vertices A, B and F. Since the branch EF is oriented from E to F, we let X be the part containing E, and Y be the part containing F. We now list all the edges of the oriented graph joining a vertex in X and a vertex in Y — these are the edges EF, BE and BC. One of these edges is the branch EF that we removed to disconnect the tree; the others are the chords BE and BC; together they form a cutset of the network graph.

We repeat this procedure for each branch of the spanning tree. The resulting cutsets are called *fundamental cutsets*; since there are $n - 1$ branches in the spanning tree, there are $n - 1$ fundamental cutsets. This leads to the following definitions.

> ### Definitions
>
> Let G be a connected graph, and let T be a spanning tree of G. The **set of fundamental cutsets** associated with T consists of the cutsets of G obtained by removing a branch of T, thus separating the tree into two parts, X and Y, and listing the edges of G joining a vertex in X and a vertex in Y. The corresponding current law equations are called the **fundamental cutset equations.**

We illustrate this definition by our example, using the same spanning tree as before.

branch	vertices in X	vertices in Y	fundamental cutset	diagram	fundamental cutset equation
AB	A, C, D, E, F	B	$\{AB, BC, BE\}$		$i_3 - i_5 - i_6 = 0$
AF	A, B	C, D, E, F	$\{AF, BC, BE, FA\}$		$i_2 + i_5 + i_6 - i_1 = 0$
CE	C	A, B, D, E, F	$\{BC, CD, CE\}$		$i_7 + i_9 - i_6 = 0$
DE	D	A, B, C, E, F	$\{CD, DE\}$		$i_8 - i_9 = 0$
EF	C, D, E	A, B, F	$\{BC, BE, EF\}$		$i_4 - i_5 - i_6 = 0$

We say that a number of cutsets in a graph G are **linearly independent** if the corresponding current law equations are linearly independent. Thus the cutsets in a set of fundamental cutsets are all linearly independent, since each cutset contains one edge (the branch of the spanning tree) not contained in any of the other cutsets. If G has n vertices, then any spanning tree has $n - 1$ branches, and so there are $n - 1$ cutsets in a fundamental set. This number is the largest number of independent cutsets in G, and is called the **cutset rank** of G. It follows that there are $n - 1$ fundamental cutset equations, and that this is the largest possible number of linearly independent cutset equations. Moreover, these equations, rather than the equations obtained by applying Kirchhoff's current law

at the vertices of G, are the appropriate equations for our purposes. We conclude that, in any electrical network problem, we can obtain all the information contained in Kirchhoff's current law by:

- choosing a spanning tree;
- finding the fundamental cutsets associated with it;
- finding the corresponding fundamental cutset equations.

An analogous result holds for the information contained in the generalized form of Kirchhoff's current law for other types of network.

Problem 2.5

For each of the spanning trees shown below, write down:

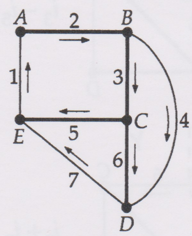
(a) a list of the branches;

(b) the corresponding sets X and Y;

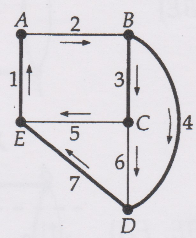
(c) the corresponding fundamental cutsets;

(d) the corresponding fundamental cutset equations.

In each case, show how to obtain the current law equation for the cutset $\{AB, BC, BD\}$ from the fundamental cutset equations.



(a)



(b)

As with fundamental cycles, we can represent the fundamental cutset equations in matrix form. The rows of this matrix correspond to the branches of a spanning tree, and the columns correspond to all the edges of the graph G. If G has n vertices and m edges, we get an $(n-1) \times m$ matrix. This matrix is called the **fundamental cutset matrix**, and is denoted by $\mathbf{D}_f$.

As we said earlier, it is usual to write the columns corresponding to the branches first, followed by those for the chords.

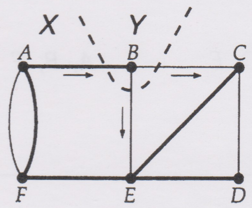
To construct this matrix, we take each branch of the spanning tree in turn, and look at the edges in the corresponding fundamental cutset. We then fill in the row corresponding to the given branch as follows: we write

1 in each column corresponding to an edge of the cutset oriented from X to Y;

−1 in each column corresponding to an edge of the cutset oriented from Y to X;

0 in each column corresponding to an edge not in the cutset.

For example, for the fundamental cutset $\{AB, BC, BE\}$ illustrated in the margin, the row corresponding to the branch AB has 1 in the column corresponding to AB, −1 in the columns corresponding to BC and BE, and 0 elsewhere.



Repeating this procedure for all five fundamental cutsets (shown on page 31), we obtain the following matrix:

$$
\begin{array}{c}
 \\ AB \\ AF \\ CE \\ DE \\ EF
\end{array}
\begin{array}{c}
\begin{array}{ccccccccc} AB & AF & CE & DE & EF & BC & BE & CD & FA \end{array} \\
\left[\begin{array}{ccccc|cccc}
1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0
\end{array}\right] \\
\underbrace{\qquad\qquad\qquad}_{\text{branches}} \quad \underbrace{\qquad\qquad\qquad}_{\text{chords}}
\end{array}
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\end{array}}\right\} \text{branches}
$$

Having found the fundamental cutset matrix $\mathbf{D}_f$, we can write the fundamental cutset equations in matrix form as

$$\mathbf{D}_f\,\mathbf{i} = \mathbf{0},$$

where $\mathbf{i}$ is the column vector of edge currents, written in the same order as the columns of the matrix. For example, the above matrix gives

$$\begin{array}{l} \text{edge number} \rightarrow \quad 3 \quad 2 \quad 7 \quad 8 \quad 4 \quad 6 \quad 5 \quad 9 \quad 1 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_3 \\ i_2 \\ i_7 \\ i_8 \\ i_4 \\ i_6 \\ i_5 \\ i_9 \\ i_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Multiplying the two matrices on the left, we obtain the following equation, which is equivalent to the five equations on page 31.

$$\begin{bmatrix} i_3 - i_6 - i_5 \\ i_2 + i_6 + i_5 - i_1 \\ i_7 - i_6 + i_9 \\ i_8 - i_9 \\ i_4 - i_6 - i_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Problem 2.6

For each of the spanning trees in Problem 2.5, find the corresponding fundamental cutset matrix, and use it to derive the fundamental cutset equations.

2.5 Obtaining the fundamental cycle and cutset matrices

So far we have shown that:

- all the information which can be obtained by applying Kirchhoff's voltage law to an electrical network is contained in the matrix equation
  $$\mathbf{C}_f\,\mathbf{v} = \mathbf{0},$$
  where $\mathbf{C}_f$ is the fundamental cycle matrix;
- all the information which can be obtained by applying Kirchhoff's current law to an electrical network is contained in the matrix equation
  $$\mathbf{D}_f\,\mathbf{i} = \mathbf{0},$$
  where $\mathbf{D}_f$ is the fundamental cutset matrix.

We have shown how the fundamental cycle matrix $\mathbf{C}_f$ and fundamental cutset matrix $\mathbf{D}_f$ with respect to a given spanning tree can be obtained by inspection of the oriented graph of an electrical network. Even for the small networks we have considered, this is a somewhat tedious process, so we would like to be able to get a computer to do this for us. The problem with this is that the method we have described is not very suitable for adaptation as a computer program. Fortunately, however, the matrices $\mathbf{C}_f$ and $\mathbf{D}_f$ can be obtained directly from the incidence matrix of the oriented graph by an elegant method which is suitable for translation into a computer program. We now describe this method, beginning with the definition of the incidence matrix of an oriented graph.

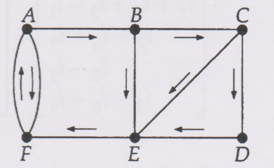
The incidence matrix of an oriented graph

We define the incidence matrix of an oriented graph by regarding the oriented graph as a digraph in which the direction of each arc is just the reference direction of the corresponding edge of the oriented graph. Adapting the definition of the incidence matrix of a digraph given in *Graphs 1*, we obtain the following definition.

> ### Definition
>
> Let G be an oriented graph with n vertices and m edges. The **incidence matrix B**(G) is the $n \times m$ matrix in which the entry in row i and column j is
>
> 1 if edge j is incident with, and oriented away from, vertex i;
>
> –1 if edge j is incident with, and oriented towards, vertex i;
>
> 0 if edge j is not incident with vertex i.

For our example, the incidence matrix is

$$\begin{array}{cc}
 & \begin{array}{ccccccccc} AB & AF & BC & BE & CD & CE & DE & EF & FA \end{array} \\
\begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \end{array} &
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & -1 & -1 & 1 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1
\end{bmatrix}
\end{array}$$



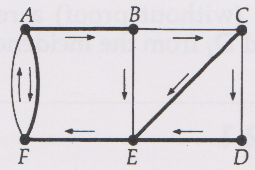
Each column of the incidence matrix contains one 1 and one –1, so that if the six rows are added together, the result is a row of zeros. Thus each row is equal to minus the sum of all the other rows. This result holds for the incidence matrix of any oriented graph. Since the rows of a matrix are linearly dependent if one of them can be obtained by adding or subtracting multiples of the others, the rows of any incidence matrix are linearly dependent. It follows that we can omit any single row of **B**(G) without loss of information. For example, in the above matrix, if we omit the row corresponding to the vertex C, then we get the following matrix. We denote it by $\mathbf{B}_0$:

$$\begin{array}{cc}
 & \begin{array}{ccccccccc} AB & AF & BC & BE & CD & CE & DE & EF & FA \end{array} \\
\begin{array}{c} A \\ B \\ D \\ E \\ F \end{array} &
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & -1 & -1 & 1 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1
\end{bmatrix}
\end{array} = \mathbf{B}_0$$

Such a matrix is called a **reduced incidence matrix**, and the vertex C is called the **reference vertex**.

If we are given the matrix $\mathbf{B}_0$, we can recover the original incidence matrix **B**(G) by inserting the missing row (calculated by taking minus the sum of the rows of $\mathbf{B}_0$). However, we cannot replace *two* missing rows, since we do not know which direction to assign to an edge joining the corresponding two vertices. It follows that if G has n vertices, then the maximum number of linearly independent rows of the incidence matrix **B**(G) is $n - 1$.

Suppose that we have chosen the spanning tree of G with branches AB, AF, CE, DE and EF. We can rearrange the columns of the reduced incidence matrix $\mathbf{B}_0$ so that those columns corresponding to branches of the tree come first, followed by those columns corresponding to the chords, as follows.



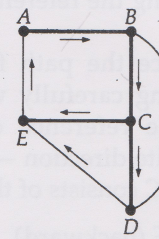
$$\mathbf{B}_0 = \begin{array}{c} \\ A \\ B \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{ccccccccc} AB & AF & CE & DE & EF & BC & BE & CD & FA \end{array} \\ \left[\begin{array}{rrrrr:rrrr} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \\ \underbrace{\hphantom{AB\;AF\;CE\;DE\;EF}}_{\text{branches}} \;\; \underbrace{\hphantom{BC\;BE\;CD\;FA}}_{\text{chords}} \end{array}$$

We have partitioned $\mathbf{B}_0$ into two parts, one part corresponding to the branches, and one part corresponding to the chords. We can abbreviate this partitioned matrix by writing $\mathbf{B}_0 = [\mathbf{B}_t \mid \mathbf{B}_c]$, where $\mathbf{B}_t$ (the tree part) consists of the first five columns, and $\mathbf{B}_c$ (the co-tree part) consists of the last four columns:

$$\mathbf{B}_t = \begin{array}{c} \\ A \\ B \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{ccccc} AB & AF & CE & DE & EF \end{array} \\ \left[\begin{array}{rrrrr} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 \\ 0 & -1 & 0 & 0 & -1 \end{array}\right] \end{array} \quad \text{and } \mathbf{B}_c = \begin{array}{c} \\ A \\ B \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{cccc} BC & BE & CD & FA \end{array} \\ \left[\begin{array}{rrrr} 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

Problem 2.7

Consider the oriented graph and spanning tree shown in the margin.



Write down:

(a) the incidence matrix $\mathbf{B}$ of G;

(b) the reduced incidence matrix $\mathbf{B}_0$, using vertex D as reference vertex;

(c) the matrices $\mathbf{B}_t$ and $\mathbf{B}_c$, using the spanning tree shown.

Obtaining the fundamental cycle and cutset matrices from the incidence matrix

The reduced incidence matrix $\mathbf{B}_0$ completely specifies the corresponding oriented graph — that is, it tells us which edges join which vertices, and it tells us the reference direction of each edge. It is a convenient representation of the graph to use for computer analysis, since it can easily be written down by inspection of the oriented graph, and it is suitable for feeding into a computer. Because the reduced incidence matrix completely describes the oriented graph, it should theoretically be possible, once a spanning tree has been specified, for the computer to generate the fundamental cutset and cycle matrices without further reference to the graph itself. We now describe how this can be done.

First we must specify a spanning tree. Any spanning tree will serve our purpose, and an appropriate algorithm (such as the one given on page 24) can be used to find a spanning tree using the information contained in the reduced incidence matrix. We can then arrange the reduced incidence matrix in the form $\mathbf{B}_0 = [\mathbf{B}_t \mid \mathbf{B}_c]$, as described above.

Suppose that the oriented graph G has n vertices; then $\mathbf{B}_t$ has $n - 1$ linearly independent rows (corresponding to the $n - 1$ linearly independent rows of the reduced incidence matrix) and $n - 1$ columns (corresponding to the $n - 1$ edges in the spanning tree). So $\mathbf{B}_t$ is a square matrix whose rows are linearly independent. It follows from the theory of matrices that $\mathbf{B}_t$ has an *inverse matrix;* that is, there exists a matrix $\mathbf{B}_t^{-1}$ of order $(n - 1) \times (n - 1)$ such that

$$\mathbf{B}_t^{-1}\mathbf{B}_t = \mathbf{B}_t\,\mathbf{B}_t^{-1} = \mathbf{I}_{n-1}$$

where $\mathbf{I}_{n-1}$ is the identity matrix of order $(n - 1) \times (n - 1)$.

We now state (without proof) a result which enables us to obtain the matrices $\mathbf{C}_f$ and $\mathbf{D}_f$ from the incidence matrix.

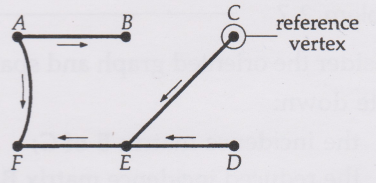
> **Theorem 2.1**
>
> The fundamental cycle matrix $\mathbf{C}_f$ and the fundamental cutset matrix $\mathbf{D}_f$ can be expressed in terms of $\mathbf{B}_t$ and $\mathbf{B}_c$ as follows:
>
> fundamental cycle matrix $\quad \mathbf{C}_f = [-(\mathbf{B}_t^{-1}\mathbf{B}_c)^T \mid \mathbf{I}_{m-n+1}]$
>
> fundamental cutset matrix $\quad \mathbf{D}_f = [\mathbf{I}_{n-1} \mid \mathbf{B}_t^{-1}\mathbf{B}_c]$.

$(\mathbf{B}_t^{-1}\mathbf{B}_c)^T$ is the transpose of $\mathbf{B}_t^{-1}\mathbf{B}_c$.

We shall explain in detail how these expressions are used to calculate $\mathbf{C}_f$ and $\mathbf{D}_f$, illustrating the procedure using our example. Unfortunately, the proofs of the validity of these expressions for $\mathbf{C}_f$ and $\mathbf{D}_f$ are too complicated to be included here.

We first turn to the problem of finding $\mathbf{B}_t^{-1}$. In general, it is not easy to find an inverse matrix. However, for this particular type of matrix $\mathbf{B}_t$, there is a simple method for finding its inverse matrix $\mathbf{B}_t^{-1}$.

STEP 1 We first draw the spanning tree, marking in the orientations of the edges, and indicating the reference vertex.



STEP 2 We next trace the path from each vertex of the tree to the reference vertex, noting carefully which branches are traversed in the same direction as the reference direction, and which branches are traversed in the opposite direction — for example, the path from vertex A to the reference vertex C consists of the branches

AF (forward), *EF* (backward), *CE* (backward).

STEP 3 Finally, we construct the matrix whose columns correspond to the vertices (other than the reference vertex) and whose rows correspond to the branches of the spanning tree. Each column represents the path from the corresponding vertex to the reference vertex; the element appearing in the column is

1 if the corresponding branch is traversed in the same direction as the reference direction;

−1 if the branch is traversed in the opposite direction;

0 if the branch is not included in the path.

For the above example, we obtain the following matrix.

$$\begin{array}{c} \\ AB \\ AF \\ CE \\ DE \\ EF \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & D & E & F \end{array} \\ \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 \end{bmatrix} \end{array}$$

This is the required matrix $\mathbf{B}_t^{-1}$. We can check this by forming the matrix products $\mathbf{B}_t^{-1}\mathbf{B}_t$ and $\mathbf{B}_t\mathbf{B}_t^{-1}$:

$$\mathbf{B}_t^{-1}\mathbf{B}_t = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 \\ 0 & -1 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{I}_5$$

$$\mathbf{B}_t\mathbf{B}_t^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 \\ 0 & -1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{I}_5.$$

Problem 2.8

Use the above method to find the inverse $\mathbf{B}_t^{-1}$ of the matrix $\mathbf{B}_t$ in Problem 2.7, and check your answer by calculating $\mathbf{B}_t^{-1}\mathbf{B}_t$ and $\mathbf{B}_t\mathbf{B}_t^{-1}$.

Having constructed $\mathbf{B}_t^{-1}$, we can use it to calculate the fundamental cutset and cycle matrices using the expressions given in Theorem 2.1. We illustrate this procedure by applying it to our example.

First we form the matrix product $\mathbf{B}_t^{-1}\mathbf{B}_c$:

$$\mathbf{B}_t^{-1}\mathbf{B}_c = \begin{array}{c} AB \\ AF \\ CE \\ DE \\ EF \end{array} \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 \end{bmatrix} \begin{array}{c} \begin{array}{cccc} BC & BE & CD & FA \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$= \begin{array}{c} AB \\ AF \\ CE \\ DE \\ EF \end{array} \begin{array}{c} \begin{array}{cccc} BC & BE & CD & FA \end{array} \\ \begin{bmatrix} -1 & -1 & 0 & 0 \\ 1 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 \end{bmatrix} \end{array}$$

We now insert the rows and columns of the identity matrix $\mathbf{I}_5$ at the left-hand side of the matrix:

$$\left[\mathbf{I}_5 \middle| \mathbf{B}_t^{-1}\mathbf{B}_c\right] = \begin{array}{c} AB \\ AF \\ CE \\ DE \\ EF \end{array} \begin{array}{c} \begin{array}{ccccc|cccc} AB & AF & CE & DE & EF & BC & BE & CD & FA \end{array} \\ \left[\begin{array}{ccccc|cccc} 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 \end{array}\right] \end{array}$$

This is the fundamental cutset matrix $\mathbf{D}_f$, as can be seen by comparing it with the matrix obtained on page 32 by inspection of the oriented graph.

To form the fundamental cycle matrix $\mathbf{C}_f$, we take the transpose of $\mathbf{B}_t^{-1}\mathbf{B}_c$:

$$\left(\mathbf{B}_t^{-1}\mathbf{B}_c\right)^T = \begin{array}{c} BC \\ BE \\ CD \\ FA \end{array} \begin{array}{c} \begin{array}{ccccc} AB & AF & CE & DE & EF \end{array} \\ \begin{bmatrix} -1 & 1 & -1 & 0 & -1 \\ -1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

We then multiply each element of this matrix by -1, and write the identity matrix $\mathbf{I}_4$ to the right of it:

$$\left[-\left(\mathbf{B}_t^{-1}\mathbf{B}_c\right)^T \middle| \mathbf{I}_4\right] = \begin{array}{c} BC \\ BE \\ CD \\ FA \end{array} \begin{array}{c} \begin{array}{ccccc|cccc} AB & AF & CE & DE & EF & BC & BE & CD & FA \end{array} \\ \left[\begin{array}{ccccc|cccc} 1 & -1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

This is just the fundamental cycle matrix $\mathbf{C}_f$, as can be seen by comparing it with the matrix we obtained on page 28 by inspection of the oriented graph.

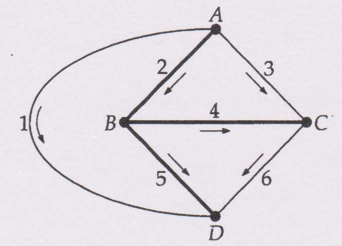
Problem 2.9

(a) Use the matrices $\mathbf{B}_c$ and $\mathbf{B}_t^{-1}$ calculated in Problems 2.7 and 2.8 to find the corresponding fundamental cutset and cycle matrices $\mathbf{D}_f$ and $\mathbf{C}_f$, and check that your answers agree with those of Problems 2.6 and 2.4 respectively.

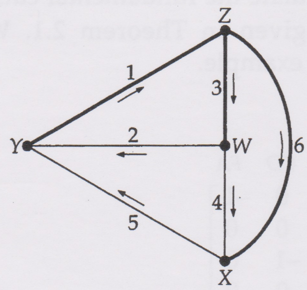
(b) Calculate the matrix product $\mathbf{C}_f\mathbf{D}_f^T$.

Problem 2.10

For each of the following oriented graphs, find the fundamental cycle equations and the fundamental cutset equations, and comment on your results. In each case, use the spanning tree indicated by *thick* edges.



(a)



(b)

2.6 Tellegen's theorem

Not Assessed

We conclude this section by showing how to prove a result of great theoretical importance, known as *Tellegen's theorem*, by making use of the incidence matrix. It may be stated in the following form.

> **Theorem 2.2: Tellegen's theorem**
>
> Suppose that there are two electrical networks which can be represented by the same oriented graph with n vertices and m edges. Then
>
> $$\sum_{k=1}^{m} v_k i_k = 0,$$
>
> where v_k is the voltage associated with the kth edge for the first network, and i_k is the current associated with the kth edge for the second network.

$$\sum_{k=1}^{m} v_k i_k = v_1 i_1 + v_2 i_2 + \ldots + v_m i_m.$$

Proof

We start by applying Kirchhoff's current law to each of the n vertices of the oriented graph. It is not difficult to see that the resulting equations can be expressed in the matrix form:

$$\mathbf{BJ} = \mathbf{0},$$

where

- $\mathbf{B}$ is the $n \times m$ incidence matrix of the oriented graph;
- $\mathbf{J}$ is the $m \times 1$ column vector of currents in the second network associated with the m edges of the graph;
- $\mathbf{0}$ is the $n \times 1$ zero vector.

We now turn to Kirchhoff's voltage law. Since the voltages associated with the edges of the graph satisfy Kirchhoff's law, we can assign a unique potential to each of the n vertices such that the voltage v_k in the first network associated with edge k is given by

$v_k = P_a - P_b,$

where P_a and P_b are the potentials at the two vertices joined by the kth edge. The equation analogous to the above for all the vertices of the graph can be expressed in the matrix form:

$$\mathbf{B}^T\mathbf{P} = \mathbf{V},$$

where

$\mathbf{B}^T$ is the $m \times n$ transpose of the incidence matrix $\mathbf{B}$;

$\mathbf{P}$ is the $n \times 1$ column vector of potentials at the n vertices;

$\mathbf{V}$ is the $m \times 1$ column vector of voltages (or potential differences) associated with the m edges of the graph.

We can write the expression in Tellegen's theorem in matrix form thus:

$$\left[\sum_{k=1}^{m} v_k i_k\right] = \mathbf{V}^T\mathbf{J},$$

The brackets are inserted to indicate that the expression on the left is a 1×1 matrix.

where $\mathbf{V}^T$ has order $1 \times m$, and $\mathbf{J}$ has order $m \times 1$, so $\mathbf{V}^T\mathbf{J}$ has order 1×1.

We know that $\mathbf{V} = \mathbf{B}^T\mathbf{P}$, so we substitute this in the above matrix expression and thus obtain:

$$\begin{aligned}\left[\sum_{k=1}^{m} v_k i_k\right] &= (\mathbf{B}^T\mathbf{P})^T\mathbf{J} \\ &= (\mathbf{P}^T\mathbf{B})\,\mathbf{J} && ((\mathbf{B}^T\mathbf{P})^T = \mathbf{P}^T(\mathbf{B}^T)^T = \mathbf{P}^T\mathbf{B}) \\ &= \mathbf{P}^T(\mathbf{B}\mathbf{J}) && \text{(property of matrix multiplication)} \\ &= \mathbf{P}^T\mathbf{0}_{n\times 1} && (\mathbf{B}\mathbf{J} = \mathbf{0}_{n\times 1}) \\ &= \mathbf{0}_{1\times 1} && (\mathbf{P}^T \text{has order } 1 \times n).\end{aligned}$$

Hence $\sum_{k=1}^{m} v_k i_k = 0$, as required. ■

Tellegen's theorem holds for any two networks which have the same oriented graph. Equally, we can apply it to just one network, in which case the result is that $\sum_{\text{all edges}} v_k i_k = 0$, where v_k and i_k are the voltage and current associated with the kth edge of this network. The product $v_k i_k$ is the power (or rate of change of energy) associated with the kth edge, so the equation $\sum_{\text{all edges}} v_k i_k = 0$ tells us that energy is conserved in the network.

This is the law of conservation of energy for a network.

Note that we have derived this well-known law of physics for an electrical network from Kirchhoff's current and voltage laws. In fact, given any two of these three laws (the conservation of energy law, Kirchhoff's current law and Kirchhoff's voltage law), it is possible to derive the third.

Although we have stated Tellegen's theorem for electrical networks, it also holds for the through and across variables of any physical systems which obey generalized forms of Kirchhoff's laws.

After studying this section, you should be able to:

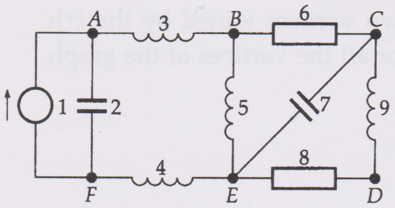
- write down the voltage and current law equations for a given network or oriented graph;
- find the *fundamental cycles* and the *fundamental cycle matrix* corresponding to a given spanning tree;
- find the *fundamental cutsets* and the *fundamental cutset matrix* corresponding to a given spanning tree;
- show how the fundamental cutset and cycle matrices can be obtained from the incidence matrix of an oriented graph.

Summary

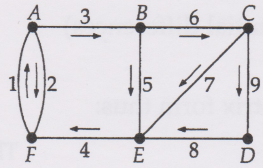
On the next two pages, we summarize what we have learned about the two examples of this section. These results will be used in the next section.

Example in text

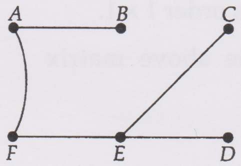
network



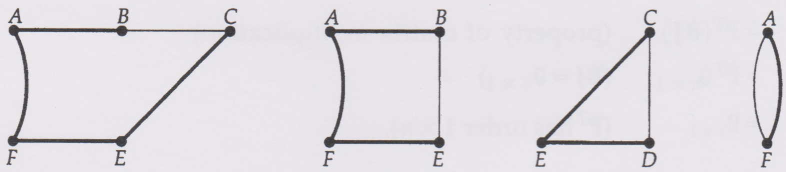
oriented graph



spanning tree



fundamental cycles



fundamental cycle equations

$v_3 + v_6 + v_7 + v_4 - v_2 = 0, \quad v_3 + v_5 + v_4 - v_2 = 0, \quad v_9 + v_8 - v_7 = 0, \quad v_2 + v_1 = 0$

fundamental cycle matrix

$$\mathbf{C}_f = \begin{array}{cc} & \begin{matrix} AB & AF & CE & DE & EF & BC & BE & CD & FA \end{matrix} \\ \begin{matrix} BC \\ BE \\ CD \\ FA \end{matrix} & \left[\begin{array}{rrrrr:rrrr} 1 & -1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]\left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\}\text{chords} \\ & \underbrace{\hphantom{AB\quad AF\quad CE\quad DE\quad EF}}_{\text{branches}}\;\underbrace{\hphantom{BC\quad BE\quad CD\quad FA}}_{\text{chords}} \end{array}$$

fundamental cutsets

$\{AB, BC, BE\}, \{AF, BC, BE, FA\}, \{BC, CD, CE\}, \{CD, DE\}, \{BC, BE, EF\}$

fundamental cutset equations

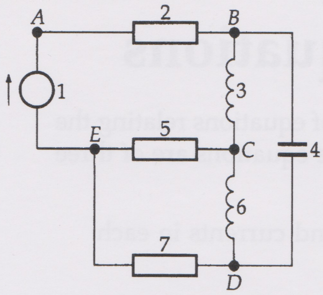
$i_3 - i_5 - i_6 = 0, \quad i_2 + i_6 + i_5 - i_1 = 0, \quad i_7 - i_6 + i_9 = 0, \quad i_8 - i_9 = 0, \quad i_4 - i_6 - i_5 = 0$

fundamental cutset matrix

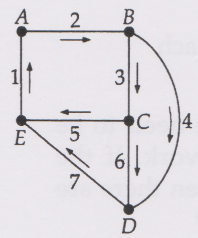
$$\mathbf{D}_f = \begin{array}{cc} & \begin{matrix} AB & AF & CE & DE & EF & BC & BE & CD & FA \end{matrix} \\ \begin{matrix} AB \\ AF \\ CE \\ DE \\ EF \end{matrix} & \left[\begin{array}{rrrrr:rrrr} 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 \end{array}\right]\left.\vphantom{\begin{matrix}1\\1\\1\\1\\1\end{matrix}}\right\}\text{branches} \\ & \underbrace{\hphantom{AB\quad AF\quad CE\quad DE\quad EF}}_{\text{branches}}\;\underbrace{\hphantom{BC\quad BE\quad CD\quad FA}}_{\text{chords}} \end{array}$$

Example in problems

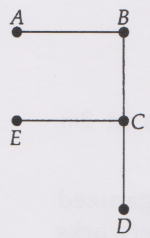
network



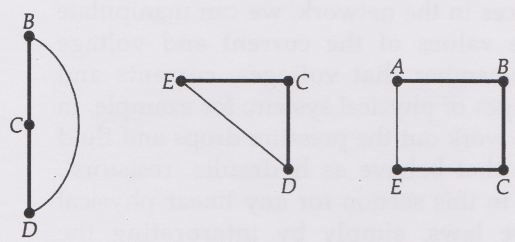
oriented graph



spanning tree



fundamental cycles



fundamental cycle equations

$v_4 - v_6 - v_3 = 0, \quad v_6 + v_7 - v_5 = 0, \quad v_2 + v_3 + v_5 + v_1 = 0$

fundamental cycle matrix

$$\mathbf{C}_\mathrm{f} = \begin{array}{c} \\ BD \\ DE \\ EA \end{array} \begin{array}{c} \begin{array}{ccccccc} AB & BC & CD & CE & BD & DE & EA \end{array} \\ \left[\begin{array}{cccc|ccc} 0 & -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \\ \begin{array}{cc} \underbrace{\qquad\qquad\qquad}_{\text{branches}} & \underbrace{\qquad\quad}_{\text{chords}} \end{array} \end{array} \left.\vphantom{\begin{array}{c}0\\0\\0\end{array}}\right\} \text{chords}$$

fundamental cutsets

$\{AB, EA\}, \{BC, BD, EA\}, \{BD, CD, DE\}, \{CE, DE, EA\}$

fundamental cutset equations

$i_2 - i_1 = 0, \quad i_3 + i_4 - i_1 = 0, \quad i_6 + i_4 - i_7 = 0, \quad i_5 + i_7 - i_1 = 0$

fundamental cutset matrix

$$\mathbf{D}_\mathrm{f} = \begin{array}{c} \\ AB \\ BC \\ CD \\ CE \end{array} \begin{array}{c} \begin{array}{ccccccc} AB & BC & CD & CE & BD & DE & EA \end{array} \\ \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right] \\ \begin{array}{cc} \underbrace{\qquad\qquad\qquad}_{\text{branches}} & \underbrace{\qquad\quad}_{\text{chords}} \end{array} \end{array} \left.\vphantom{\begin{array}{c}0\\0\\0\\0\end{array}}\right\} \text{branches}$$

3 Electrical networks: solving the network equations

In the previous two sections we obtained a number of equations relating the voltages and currents in an electrical network. These equations are of three types:

1. **component equations**, relating the voltages and currents in each component;
2. **fundamental cycle equations**, relating the voltages in each fundamental cycle of the network;
3. **fundamental cutset equations**, relating the currents in each fundamental cutset of the network.

These three sets of equations contain all the information we need to be able to work out all the voltages and currents in the network. If the oriented graph of the network has n vertices and m edges, then there are m voltages and m currents — a total of $2m$ unknowns related by

1. m component equations;
2. $m - n + 1$ fundamental cycle equations;
3. $n - 1$ fundamental cutset equations.

There are, therefore, $m + (m - n + 1) + (n - 1) = 2m$ equations in $2m$ unknowns.

In this section we show how these $2m$ equations can be organized systematically into a standard matrix form. We look first at networks that contain only resistors and independent sources, and show that, given the values of the independent sources in the network, we can manipulate the matrix equations to find the values of the current and voltage associated with each resistor. Remember that voltages, currents and resistors have analogues in other types of physical system; for example, in a hydraulic system we may wish to work out the pressure drops and fluid flow rates associated with valves that behave as hydraulic 'resistors'. We can use the method described in this section for any linear physical system that obeys Kirchhoff-type laws, simply by interpreting the currents and voltages as the through and across variables of the system under consideration.

The graphical approach we describe for the analysis of physical networks involves the construction and manipulation of large matrices — generally of order $2m \times 2m$ for a network whose oriented graph has m edges. So, for example, when using this method to work out all the voltages and currents in an electrical network containing only three components (say a battery and two resistors), we need to handle a 6×6 matrix. The solution procedure, therefore, is not particularly suited to hand calculation, but can be relatively easily programmed on a computer. We discuss a method called *Gaussian elimination*, which is a systematic procedure for solving a set of simultaneous equations in matrix form and which forms the basis of a number of computer solution algorithms.

In Section 3.3, we extend our discussion to electrical networks that contain capacitors and inductors as well as resistors and independent sources. For such a network, the component equations involve not only v and i, but also their derivatives with respect to time. We deal with this type of equation by specifying certain of the voltages and currents as *state variables*. We then set up equations, called *state equations*, involving the state variables and their derivatives. We solve these equations for the state variables and show that the remaining network variables can be expressed in terms of these state variables.

3.1 Formulation of the matrix equation

We first consider the component matrix, and then show how to combine it with the fundamental cycle and cutset matrices.

The component matrix

Our first task is to write down the component equations of our network in matrix form. In Section 1 we discussed four types of component — resistors, capacitors, inductors and independent sources. For the moment, however, we restrict our attention to networks that contain only resistors and independent sources. Later, we discuss the modifications that must be made when capacitors and/or inductors are present.

Now the component equations corresponding to the resistors are all of the form

$v = Ri,$

and the component equations corresponding to the independent sources have the form

$v = V$ for a voltage source,

or

$i = I$ for a current source.

We begin by rewriting the component equations so that all the through and across variables appear on the left-hand side, and all the terms not involving these variables appear on the right, thus:

$1v - Ri = 0$ for a resistor;

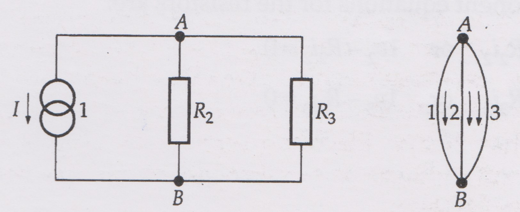
$1v + 0i = V$ for a voltage source;

$0v + 1i = I$ for a current source.

Note that we have included the coefficients 0 and 1 where appropriate — this is because our next step is to rewrite the component equations in matrix form. We illustrate this procedure by two examples.

Example 3.1

Consider the following network and its oriented graph:



The component equations for the resistors are:

$$v_2 = R_2 i_2$$
$$v_3 = R_3 i_3$$

which we can write as

$$1v_2 - R_2 i_2 = 0$$
$$1v_3 - R_3 i_3 = 0.$$

We can write these equations in matrix form as follows:

$$\left[\begin{array}{cc|cc} 1 & 0 & -R_2 & 0 \\ 0 & 1 & 0 & -R_3 \end{array}\right] \left[\begin{array}{c} v_2 \\ v_3 \\ \hline i_2 \\ i_3 \end{array}\right] = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (3.1)$$

There are 2 resistors, so on the left we have a 2×4 matrix of coefficients, multiplied by a 4×1 column vector of edge variables, and on the right we have the 2×1 zero column vector.

There is only one independent (current) source I, so we have only one further component equation,

$i_1 = I,$

which we can write as

$0v_1 + 1i_1 = I.$

We can write this equation in matrix form as follows:

$$[0 \;\; 1]\begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = [I] \tag{3.2}$$

We now combine the matrix equations (3.1) and (3.2). To do this, we incorporate all the edge variables into a single column vector **z** as follows:

$$\mathbf{z} = \begin{bmatrix} v_2 \\ v_3 \\ v_1 \\ i_2 \\ i_3 \\ i_1 \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \text{voltages} \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \text{currents} \end{matrix}$$

We list the currents in the same order as the corresponding voltages; at this stage, the order chosen is unimportant.

Thus we obtain the matrix equation

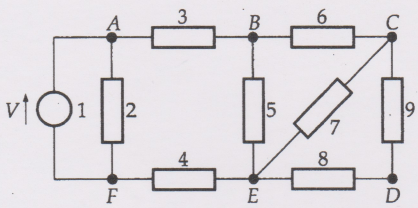
$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -R_2 & 0 & 0 \\ 0 & 1 & 0 & 0 & -R_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} v_2 \\ v_3 \\ v_1 \\ i_2 \\ i_3 \\ i_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}$$

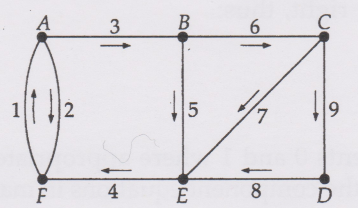
We have shaded the partitions arising from equation (3.1).

■

Example 3.2

Consider the following network and its oriented graph:





The component equations for the resistors are:

$v_2 = R_2 i_2 \quad$ or $\quad 1v_2 - R_2 i_2 = 0$

$v_3 = R_3 i_3 \quad$ or $\quad 1v_3 - R_3 i_3 = 0$

$\vdots \qquad\qquad\qquad \vdots$

$v_9 = R_9 i_9 \quad$ or $\quad 1v_9 - R_9 i_9 = 0.$

We can write these equations in matrix form as follows:

$$\left[\begin{array}{cccccccc:cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -R_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -R_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -R_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -R_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -R_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -R_7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -R_8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -R_9 \end{array}\right] \begin{bmatrix} v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \\ i_8 \\ i_9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.1}$$

There are 8 resistors, so on the left we have an 8×16 matrix of coefficients, multiplied by a 16×1 column vector of edge variables, and on the right we have the 8×1 zero column vector.

There is only one independent (voltage) source V, so we have only one further equation,

$$v_1 = -V,$$

which we can write as

$$1v_1 + 0\,i_1 = -V.$$

We can write this equation in matrix form as follows:

$$\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = [-V]. \qquad (3.2)$$

Note that the reference arrow for edge 1 of the oriented graph points in the same direction as the arrow on the voltage source in the circuit diagram. Because of the different conventions used in the two cases, the component equation has a minus sign in it.

Combining the matrix equations (3.1) and (3.2) as before, we obtain the matrix equation

$$\left[\begin{array}{ccccccccc|ccccccccc}
1&0&0&0&0&0&0&0&0&-R_2&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&-R_3&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&-R_4&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&-R_5&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&-R_6&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&-R_7&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&-R_8&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&-R_9&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0
\end{array}\right]
\begin{bmatrix} v_2\\ v_3\\ v_4\\ v_5\\ v_6\\ v_7\\ v_8\\ v_9\\ v_1\\ \hline i_2\\ i_3\\ i_4\\ i_5\\ i_6\\ i_7\\ i_8\\ i_9\\ i_1 \end{bmatrix}
=
\begin{bmatrix} 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ -V \end{bmatrix}$$

The above examples show how we deal with resistors, independent current sources, and independent voltage sources. Note that, in each example, the matrix of coefficients on the left is very simple — most of the elements are zero, and each non-zero element is either 1 or $-R_j$, for some resistance R_j.

More generally, for a given network of this type for which the oriented graph has m edges, we write down all the component equations and then rearrange them in matrix form

$$\mathbf{T}\begin{bmatrix} \mathbf{v} \\ \hline \mathbf{i} \end{bmatrix} = \mathbf{f},$$

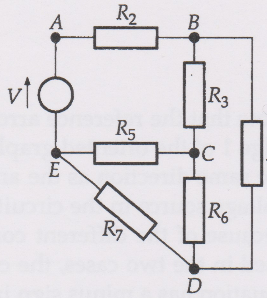
where

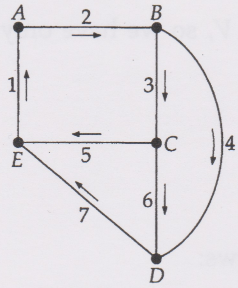
- $\mathbf{T}$ is the $m \times 2m$ matrix of coefficients which appear in the component equations;
- $\begin{bmatrix} \mathbf{v} \\ \hline \mathbf{i} \end{bmatrix}$ is the $2m \times 1$ column vector of all the voltages, followed by all the currents, listed in the same order;
- $\mathbf{f}$ is an $m \times 1$ column vector containing terms from the right-hand side of the component equations for independent sources (such as I and $-V$).

In fact, most of the elements of the matrix $\mathbf{T}$ are zero. Such a matrix is said to be *sparse*.

Problem 3.1

Consider the following network and its oriented graph:





Assuming that R_2, R_3, …, R_7 and V are known, write down the component equations in matrix form.

We now have the component equations in matrix form. These equations describe the constraints on the voltages and currents in the network that are imposed by the components only. But we know that the voltages and currents in the network must also obey Kirchhoff's laws. Our next step, therefore, is to combine the information contained in the component equations with the information about how the components are connected together which is contained in the fundamental cycle and cutset equations.

The fundamental cycle and cutset matrices

We saw in Section 2 that the fundamental cycle and cutset equations can be expressed in matrix form — namely,

$$\mathbf{C}_\mathrm{f}\,\mathbf{v} = \mathbf{0} \quad \text{and} \quad \mathbf{D}_\mathrm{f}\,\mathbf{i} = \mathbf{0},$$

where $\mathbf{C}_\mathrm{f}$ and $\mathbf{D}_\mathrm{f}$ are the fundamental cycle and cutset matrices associated with a given spanning tree, and $\mathbf{v}$ and $\mathbf{i}$ are column vectors of edge voltages and currents written in the same order as the columns of the matrix.

For example, if we take the oriented graph of the network in Example 3.2, then the matrix equations are:

$$\underbrace{\underset{\mathbf{C}\qquad\qquad\qquad\qquad\mathbf{I}_4}{\left[\begin{array}{ccccc|cccc} 1 & -1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]}}_{\mathbf{C}_\mathrm{f}} \underbrace{\left[\begin{array}{c} v_3 \\ v_2 \\ v_7 \\ v_8 \\ v_4 \\ \hdashline v_6 \\ v_5 \\ v_9 \\ v_1 \end{array}\right]}_{\mathbf{v}} = \underbrace{\left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\right]}_{\mathbf{0}}$$

The oriented graph for Example 3.2 is the same as the oriented graph for the example on page 24. We choose the same spanning tree, so these matrices are the matrices summarized on page 40.

and

$$\underbrace{\underset{\mathbf{I}_5\qquad\qquad\qquad\qquad\mathbf{D}}{\left[\begin{array}{ccccc|cccc} 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 \end{array}\right]}}_{\mathbf{D}_\mathrm{f}} \underbrace{\left[\begin{array}{c} i_3 \\ i_2 \\ i_7 \\ i_8 \\ i_4 \\ \hdashline i_6 \\ i_5 \\ i_9 \\ i_1 \end{array}\right]}_{\mathbf{i}} = \underbrace{\left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right]}_{\mathbf{0}}$$

In general, it is sometimes convenient to write these equations in abbreviated form as

$$\left[\mathbf{C}\middle|\mathbf{I}_{m-n+1}\right]\mathbf{v} = \mathbf{0},$$

where $\mathbf{C}$ is an $(m-n+1)\times(n-1)$ matrix, and

$$\left[\mathbf{I}_{n-1}\middle|\mathbf{D}\right]\mathbf{i} = \mathbf{0},$$

where $\mathbf{D}$ is an $(n-1)\times(m-n+1)$ matrix.

Note that $\mathbf{C} = -(\mathbf{B}_t^{-1}\,\mathbf{B}_c)^T$ and $\mathbf{D} = \mathbf{B}_t^{-1}\mathbf{B}_c$ in the notation of Section 2.

Combining the matrix equations

We now have three matrix equations, corresponding to the component equations, the fundamental cycles, and the fundamental cutsets. These matrix equations are:

component equation: $\mathbf{T}\begin{bmatrix}\mathbf{v}\\ \hdashline \mathbf{i}\end{bmatrix} = \mathbf{f},$

fundamental cycle equation: $\mathbf{C}_f\,\mathbf{v} = \mathbf{0},$

fundamental cutset equation: $\mathbf{D}_f\,\mathbf{i} = \mathbf{0}.$

Our aim is to solve these equations — that is, to work out the voltages $\mathbf{v}$ and currents $\mathbf{i}$ that satisfy simultaneously the component equations, Kirchhoff's voltage law and Kirchhoff's current law. Because we want to solve these equations simultaneously, we start by combining them into just one matrix equation of the form

$$\mathbf{H}\begin{bmatrix}\mathbf{v}\\ \hdashline \mathbf{i}\end{bmatrix} = \begin{bmatrix}\mathbf{f}\\ \mathbf{0}\\ \mathbf{0}\end{bmatrix}.$$

This is possible, because we can write the left-hand side of each of the three equations as a matrix multiplying a vector of voltages and currents.

The component equation is already in this form:

$$\mathbf{T}\begin{bmatrix}\mathbf{v}\\ \hdashline \mathbf{i}\end{bmatrix} = \mathbf{f}.$$

The fundamental cycle equation can be written as

$$\left[\mathbf{C}_f \;\vdots\; \mathbf{0}\right]\begin{bmatrix}\mathbf{v}\\ \hdashline \mathbf{i}\end{bmatrix} = \mathbf{0},$$

and, similarly, the fundamental cutset equation can be written as

$$\left[\mathbf{0} \;\vdots\; \mathbf{D}_f\right]\begin{bmatrix}\mathbf{v}\\ \hdashline \mathbf{i}\end{bmatrix} = \mathbf{0},$$

At this stage, a word of caution is necessary. The voltages which occur in the column vector $\mathbf{v}$ in the fundamental cycle equation appear *in a particular order* determined partly by the choice of spanning tree. The currents which occur in the column vector $\mathbf{i}$ in the fundamental cutset equation appear in the same order as the voltages in the fundamental cycle equation. It follows that if we want to combine these equations with the component equation, we must put the voltages and currents occurring in the column vector $\begin{bmatrix}\mathbf{v}\\ \hdashline \mathbf{i}\end{bmatrix}$ in the component equation into *the same order* as the voltages and currents in the other two matrix equations.

Fortunately, this can be done very easily. All we need to do is to take the resistances R_k and the zero which occur in the diagonal of the right-hand portion of $\mathbf{T}$ in the component equation, and write them in the required

order. For example, in the resistor network of Example 3.2, the voltages appear in the fundamental cycle matrix equation in the order

$$v_3, v_2, v_7, v_8, v_4, v_6, v_5, v_9, v_1,$$

and similarly for the currents in the fundamental cutset matrix equation. The resistances and the zero on the diagonal in **T** must therefore appear in the corresponding order:

$$R_3, R_2, R_7, R_8, R_4, R_6, R_5, R_9, 0.$$

Once we have written the voltages and currents in each equation in the same order, we can combine the three matrix equations into one.

We have

$$\mathbf{T}\left[\begin{array}{c}\mathbf{v}\\ \hline \mathbf{i}\end{array}\right]=\mathbf{f},$$

$$\left[\mathbf{C}_f \mid \mathbf{0}\right]\left[\begin{array}{c}\mathbf{v}\\ \hline \mathbf{i}\end{array}\right]=\mathbf{0},$$

$$\left[\mathbf{0} \mid \mathbf{D}_f\right]\left[\begin{array}{c}\mathbf{v}\\ \hline \mathbf{i}\end{array}\right]=\mathbf{0},$$

which can be written as one equation

$$\left[\begin{array}{c|c}\multicolumn{2}{c}{\mathbf{T}}\\ \hline \mathbf{C}_f & \mathbf{0}\\ \hline \mathbf{0} & \mathbf{D}_f\end{array}\right]\left[\begin{array}{c}\mathbf{v}\\ \hline \mathbf{i}\end{array}\right]=\left[\begin{array}{c}\mathbf{f}\\ \hline \mathbf{0}\\ \hline \mathbf{0}\end{array}\right]$$

or

$$\mathbf{Hx}=\mathbf{y},$$

where

- **H** is a $2m \times 2m$ matrix containing component, cycle and cutset information;
- **x** is a column vector containing all the voltages and currents;
- **y** is a column vector consisting mainly of zeros, but also including terms such as I and $-V$ (source functions).

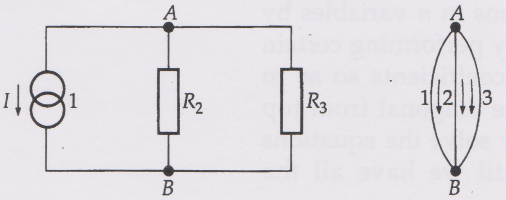
The resistor network in Example 3.2 has 9 components (8 resistors and one independent voltage source), so the matrix H corresponding to this network is an 18×18 matrix. Writing the matrix equation for the network out in full, we get the following:

$$\left[\begin{array}{ccccccccc|ccccccccc}
1&0&0&0&0&0&0&0&0&-R_3&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&-R_2&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&-R_7&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&-R_8&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&-R_4&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&-R_6&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&-R_5&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&-R_9&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
\hline
1&-1&1&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&-1&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&-1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&1&0&0&0&0&-1&-1&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&0&-1\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&-1&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&-1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&-1&-1&0&0
\end{array}\right]
\left[\begin{array}{c}v_3\\v_2\\v_7\\v_8\\v_4\\v_6\\v_5\\v_9\\v_1\\ \hline i_3\\i_2\\i_7\\i_8\\i_4\\i_6\\i_5\\i_9\\i_1\end{array}\right]
=
\left[\begin{array}{c}0\\0\\0\\0\\0\\0\\0\\0\\-V\\ \hline 0\\0\\0\\0\\ \hline 0\\0\\0\\0\\0\end{array}\right]$$

We have partitioned each of the matrices $\mathbf{C}_f$ and $\mathbf{D}_f$ into two, to reveal the identity matrices they contain.

Problem 3.2

Consider the following network and its oriented graph:



This is the network discussed earlier in Example 3.1.

Taking the left-hand edge as the spanning tree, find the fundamental cycle and cutset matrices $\mathbf{C}_f$ and $\mathbf{D}_f$. Use these, together with the matrix on page 44, to find the corresponding matrix equation $\mathbf{Hx} = \mathbf{y}$.

Problem 3.3

Use the fundamental cycle and cutset matrices given on page 41 to find the corresponding matrix equation for the network in Problem 3.1.

3.2 Solving the matrix equation

Not Assessed

The matrix $\mathbf{H}$ and the column vector $\mathbf{y}$ contain all the information we need to be able to work out the $2m$ voltages and currents in the vector $\mathbf{x}$. One approach to solving the matrix equation is to rearrange the equation so that we express $\mathbf{x}$ in terms of $\mathbf{H}$ and $\mathbf{y}$. We have

$$\mathbf{Hx} = \mathbf{y}.$$

In formulating the component, cycle and cutset matrix equations, we have taken care to ensure that the $2m$ equations in $2m$ unknowns that describe the network are linearly independent. This means that the matrix $\mathbf{H}$ has an inverse $\mathbf{H}^{-1}$. If we multiply each side of the above equation by $\mathbf{H}^{-1}$, we get

$$\mathbf{H}^{-1}\mathbf{Hx} = \mathbf{H}^{-1}\mathbf{y}.$$

Remember that the product $\mathbf{H}^{-1}\mathbf{H}$ is equal to the identity matrix $\mathbf{I}_{2m}$, so

$$\mathbf{I}_{2m}\mathbf{x} = \mathbf{H}^{-1}\mathbf{y}$$

or simply

$$\mathbf{x} = \mathbf{H}^{-1}\mathbf{y}.$$

From a theoretical point of view, the problem is now solved — in principle, we can go on to calculate $\mathbf{H}^{-1}$ for the network and hence work out each voltage and current in $\mathbf{x}$. However, even if we use a computer to perform the inversion and subsequent calculations, this is generally not the most efficient approach. One reason for this is that we are interested in $\mathbf{H}^{-1}$ only as a means to an end. What we are actually interested in are the voltages and currents in the network. So if we can work out the values of these variables without having to work out all the elements of $\mathbf{H}^{-1}$, then we can reduce the computational effort and hence the time required to analyse the network.

Instead of considering how to compute $\mathbf{H}^{-1}$, we briefly discuss a technique that forms the basis of many computer-based methods for solving a set of n linear equations in n unknowns. This technique is called *Gaussian elimination*.

Gaussian elimination

Gaussian elimination is a systematic way of solving a linearly independent system of n simultaneous linear equations in n variables by eliminating the variables one at a time. This is done by performing certain allowable operations on the rows of the matrix of coefficients so as to reduce all the elements below the main diagonal (the diagonal from top left to bottom right) to zero. We can then successively solve the equations one at a time, starting with the last equation, until we have all the required variables.

The operations we allow on the rows of the matrix are:

(a) interchange two rows;

(b) multiply any row by a non-zero number;

(c) add any row to any other row.

Note that operations (b) and (c) mean that we can add (or subtract) any multiple of any row to (or from) any other.

If we also carry out these row operations on the column vector $\mathbf{y}$, then each of them corresponds to an operation on the corresponding equations which leaves the solution of the equations unchanged. For example, operation (a) corresponds to interchanging two equations, which clearly does not affect the solution.

Example 3.3

Our aim is to describe the method of Gaussian elimination as simply and as briefly as possible. The example used is inevitably rather artificial. In practice, if solving this system of equations by hand, we would, of course, proceed differently. Moreover, when solving a system of equations by computer, we would design the program so that rounding errors are minimized.

Consider the set of equations

$$\begin{aligned} 2x_1 + 3x_2 - x_3 &= 5 \\ 4x_1 + 4x_2 - 3x_3 &= 3 \\ 2x_1 - 3x_2 + x_3 &= 1. \end{aligned}$$

These can be written in matrix form:

$$\begin{bmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$

We want to work out the values of x_1, x_2 and x_3 without inverting the 3×3 matrix. We do this by performing the allowable row operations with the aim of changing the values of all the elements of the matrix below the main diagonal to zero.

It is possible to do this, and to obtain non-zero elements on the diagonal, because the equations are linearly independent.

The result is a matrix equation of the form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix},$$

where $a_{11} \neq 0$, $a_{22} \neq 0$, $a_{33} \neq 0$.

This is equivalent to the three equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= y_1 \\ a_{22}x_2 + a_{23}x_3 &= y_2 \\ a_{33}x_3 &= y_3. \end{aligned}$$

From the last equation,

$$a_{33}x_3 = y_3 \qquad (a_{33} \neq 0)$$

we obtain

$$x_3 = \frac{y_3}{a_{33}}.$$

From the second equation,

$$a_{22}x_2 + a_{23}x_3 = y_2 \qquad (a_{22} \neq 0)$$

we obtain

$$x_2 = \frac{y_2 - a_{23}x_3}{a_{22}}.$$

Substituting $x_3 = y_3 / a_{33}$, we get

$$x_2 = \frac{y_2}{a_{22}} - \frac{a_{23}y_3}{a_{22}a_{33}}.$$

Similarly, we can work out x_1 by substituting for x_1 and x_2 in the first equation.

This method of solution is called **back-substitution** and can be extended to systems of linearly independent linear equations of any order.

Let us consider how we use Gaussian elimination to reduce the 3×3 matrix of our example to triangular form. The matrix is:

$$\begin{bmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ 2 & -3 & 1 \end{bmatrix}.$$

We start with the first column. We require that all the elements in this column except the top left be zero.

This is to eliminate the term in x_1 from the second and third equations.

If we multiply the first row by -2, we get

$$-4 \quad -6 \quad 2.$$

Adding this to the second row reduces that row to

$$0 \quad -2 \quad -1.$$

Turning our attention to the third row, we subtract the first row to give

$$0 \quad -6 \quad 2.$$

We must also remember to carry out the same operations on the vector $\begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$ on the right-hand side of the matrix equation. Thus the first element 5 of this vector is unaffected, the second element 3 becomes

$$3 + (-2 \times 5) = -7,$$

and the third element 1 becomes

$$1 - 5 = -4.$$

The matrix equation can now be written as

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & -2 & -1 \\ 0 & -6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -7 \\ -4 \end{bmatrix}$$

Our operations to modify the first column using the first row are complete. We now consider the second column and the third row. Multiplying the second row by -3 and adding the result to the third row, we reduce the third row to

$$0 \quad 0 \quad 5$$

so the second column has the required form.

The corresponding term -4 on the right-hand side becomes

$$-4 + (-3 \times -7) = 17.$$

Thus the final matrix equation is

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & -2 & -1 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -7 \\ 17 \end{bmatrix}$$

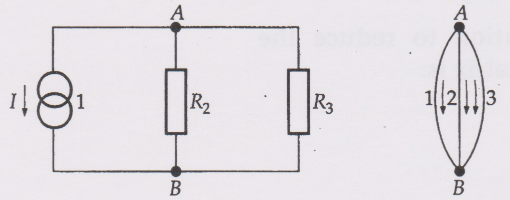
From the third equation, we find that $5x_3 = 17$, or $x_3 = 17/5$. Substituting this value for x_3 into the middle equation and solving for x_2, we get $x_2 = 9/5$. Finally, knowing x_2 and x_3, we can solve the first equation for x_1, and obtain $x_1 = 3/2$.

Using Gaussian elimination to convert the matrix to triangular form, and back-substitution, we have solved systematically the system of linear equations without having to invert the 3 × 3 matrix.

We now apply this technique to the solution to a network problem.

Worked problem

Find the currents and voltages in each part of the following network:



Solution

The matrix equation, obtained in the solution to Problem 3.2, is

$$\left[\begin{array}{ccc:ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -R_2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -R_3 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \hdashline i_1 \\ i_2 \\ i_3 \end{bmatrix}=\begin{bmatrix} I \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Interchanging row 1 and row 4, and row 5 and row 6, we get

$$\left[\begin{array}{ccc:ccc} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -R_2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -R_3 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 0 & 0 & 0 \end{array}\right]\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \hdashline i_1 \\ i_2 \\ i_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ I \\ 0 \\ 0 \end{bmatrix}.$$

Subtracting rows 1 and 3 from row 6, and row 4 from row 5, we get

$$\left[\begin{array}{ccc:ccc} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -R_2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -R_3 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 & R_3 \end{array}\right]\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \hdashline i_1 \\ i_2 \\ i_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ I \\ -I \\ 0 \end{bmatrix}.$$

Finally, after adding row 2 and $R_2 \times$ (row 5) to row 6, we get the matrix equation in the required form:

$$\left[\begin{array}{ccc:ccc} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -R_2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -R_3 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & R_2+R_3 \end{array}\right]\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \hdashline i_1 \\ i_2 \\ i_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ I \\ -I \\ -R_2 I \end{bmatrix}.$$

Successively solving these equations, beginning with the last one, we obtain

$$\begin{aligned} (R_2+R_3)i_3 &= -R_2 I, &\quad \text{so} \quad & i_3 = -R_2 I/(R_2+R_3), \\ i_2 + i_3 &= -I, && \text{so} \quad i_2 = -R_3 I/(R_2+R_3), \\ i_1 &= I, && \text{so} \quad i_1 = I, \\ v_3 - R_3 i_3 &= 0, && \text{so} \quad v_3 = -R_2 R_3 I/(R_2+R_3), \\ v_2 - R_2 i_2 &= 0, && \text{so} \quad v_2 = -R_2 R_3 I/(R_2+R_3), \\ -v_1 + v_2 &= 0, && \text{so} \quad v_1 = -R_2 R_3 I/(R_2+R_3). \end{aligned}$$

Since i_2 and i_3 are negative, the current must flow from B to A in this part of the network.

We have now achieved our goal. Given a network containing resistors and independent sources, we can work out the voltage and current associated with each edge of the oriented graph of the network.

Remark We have formulated the matrix equations so that they can be solved straightforwardly for cases in which we know the values of the source voltages or currents and wish to calculate the voltages and currents in all the other components. But suppose that we want to solve the converse problem and work out, say, the value of a source voltage which will result in a specified current flowing in one of the resistors of the network. To do this, we would have to rearrange the matrix equations. Essentially, we would have to treat the source as if it were a resistor and treat the resistor which is to have a specified current flowing in it as if it were a source. We mention this because such problems can arise in electrical network design, but we shall not discuss such a case in detail here.

All the examples we have discussed in this section have involved only 2-terminal components. However, the methods of solution we have given can be applied in a similar way to systems involving components with more than two terminals — for example, transformers or transistors. The cutset and cycle equation matrices are of the same form, but the component equation matrix is different.

3.3 State equations

Half Assessed (Matrix equation $Hx = y + K\dot{x}$ may be included and may be asked to specify the state variables.)

Up to now, we have confined our discussion to networks that contain only resistors and independent sources. For such networks, the component equations are particularly simple, being proportionality relations of the form $v = Ri$.

We now turn our attention to networks involving capacitors and inductors. For such networks, the component equations involve not only v and i, but also their derivatives. Our aim is to show how the matrix equations can be adapted to take the derivatives into account.

In Section 1 we saw that the component equations of a capacitor and an inductor are

$$i = C\frac{dv}{dt} \quad \text{and} \quad v = L\frac{di}{dt},$$

respectively, where C is the capacitance and L is the inductance of the component. Using a dot to denote a derivative with respect to time t, we may rewrite these equations in the form

$$\dot{v} = \frac{1}{C}i \quad \text{and} \quad \dot{i} = \frac{1}{L}v.$$

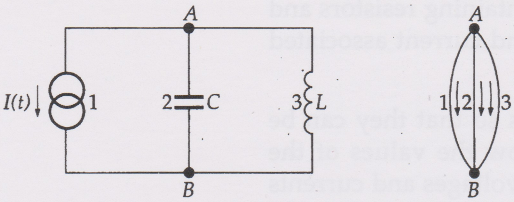
We shall use these equations to obtain the matrix equation in the form

$$\mathbf{Hx} = \mathbf{y} + \mathbf{K\dot{x}},$$

where

- $\mathbf{x}$ is a column vector involving all the voltages and currents;
- $\mathbf{\dot{x}}$ is a column vector involving their derivatives;
- $\mathbf{y}$ is the 'source function vector' we introduced earlier;
- $\mathbf{H}$ is a large matrix involving the components and the fundamental cycles and cutsets;
- $\mathbf{K}$ is a matrix isolating the derivatives of the edge variables.

We illustrate the ideas involved by considering the following simple network and its oriented graph:



The component equations for this network are

$$i_1 = I(t),$$

$$\dot{v}_2 = \frac{1}{C}\, i_2,$$

$$\dot{i}_3 = \frac{1}{L}\, v_3.$$

We write these equations in matrix form as follows:

$$\left[\begin{array}{cccc:cc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1/L & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/C & 0 & 0 \end{array}\right] \begin{bmatrix} v_1 \\ v_3 \\ i_1 \\ i_2 \\ \hdashline v_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} I(t) \\ 0 \\ 0 \end{bmatrix} + \left[\begin{array}{cccc:cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} \dot{v}_1 \\ \dot{v}_3 \\ \dot{i}_1 \\ \dot{i}_2 \\ \hdashline \dot{v}_2 \\ \dot{i}_3 \end{bmatrix}$$

Notice that the variables whose derivatives occur in the component equations (v_2 and i_3) appear *last of all* in the column vectors $\mathbf{x}$ and $\dot{\mathbf{x}}$. These variables are called **state variables**. The values of the state variables represent the *state* of the system.

We now wish to combine this equation with the fundamental cycle and cutset equations for the network. These are the same as the fundamental cycle and cutset equations for the network of Example 3.1, and can be written as follows:

$$\left[\begin{array}{ccc:ccc} -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \hdashline i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \hdashline 0 \end{bmatrix}$$

This matrix is the lower half of the matrix in the solution to Problem 3.2(a).

The shaded columns correspond to the state variables v_2 and i_3.

Rearranging the columns so as to put the edge variables into the same order as in the component equation, we get

$$\left[\begin{array}{cccc:cc} -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \begin{bmatrix} v_1 \\ v_3 \\ i_1 \\ i_2 \\ \hdashline v_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Combining this with the component matrix equation above, we obtain the required equation:

$$\underset{\mathbf{H}}{\left[\begin{array}{cccc:cc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1/L & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/C & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]} \underset{\mathbf{x}}{\begin{bmatrix} v_1 \\ v_3 \\ i_1 \\ i_2 \\ \hdashline v_2 \\ i_3 \end{bmatrix}} = \underset{\mathbf{y}}{\begin{bmatrix} I(t) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}} + \underset{\mathbf{K}}{\left[\begin{array}{cccc:cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]} \underset{\dot{\mathbf{x}}}{\begin{bmatrix} \dot{v}_1 \\ \dot{v}_3 \\ \dot{i}_1 \\ \dot{i}_2 \\ \hdashline \dot{v}_2 \\ \dot{i}_3 \end{bmatrix}}$$

$$\uparrow \qquad \uparrow \qquad \uparrow \qquad \uparrow \qquad \uparrow$$

$$\mathbf{H} \qquad \mathbf{x} = \mathbf{y} + \mathbf{K} \qquad \dot{\mathbf{x}}$$

Our next step is to manipulate these matrices so that we can write down two **state equations** for the network. The state equations are first-order differential equations that relate the derivatives of the state variables to the state variables and the independent sources. It can be shown that the two state equations are sufficient to describe completely the dynamic behaviour of this network. In other words, any voltage or current in the network may be expressed in terms of the state variables v_2 and i_3, their derivatives $\dot{v}_2$ and $\dot{i}_3$, and the independent current source $I(t)$.

To solve the matrix equations for v_2 and i_3, we use Gaussian elimination. But, because we are interested only in v_2 and i_3, we need concentrate only on the rows of the **H** matrix which involve these variables, that is, on rows 4 and 6.

Notice that row 4 has a 1 in column 5, corresponding to the 5th element in **x**, namely v_2, and that row 6 has a 1 in column 6, corresponding to i_3.

We want the final matrix equation to be of the form

$$\left[\begin{array}{cccc:cc} & ? & & & ? & \\ & & & & & \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hdashline ? & ? & ? & ? & ? & ? \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]\begin{bmatrix} v_1 \\ v_3 \\ i_1 \\ i_2 \\ v_2 \\ i_3 \end{bmatrix} = \left[\begin{array}{c} ? \\ \\ \hdashline \\ ? \end{array}\right] + \left[\begin{array}{cccc:c} & ? & & & ? \\ 0 & 0 & 0 & 0 & \\ \hdashline ? & ? & ? & ? & \\ 0 & 0 & 0 & 0 & ? \end{array}\right]\begin{bmatrix} \dot{v}_1 \\ \dot{v}_3 \\ \dot{i}_1 \\ \dot{i}_2 \\ \dot{v}_2 \\ \dot{i}_3 \end{bmatrix}$$

Those regions denoted by ? are to be determined.

We wish first to eliminate the term –1 in the first column of row 4 of **H**. We do this in two stages. First we subtract row 5 from row 4, giving the new row 4 of **H**:

$$0 \ -1 \quad 0 \quad 0 \quad 1 \quad 0.$$

A new term, –1, has now appeared in the second column, but we can eliminate that by multiplying row 2 by L, and then adding it to the new row 4, giving

$$0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0.$$

Next, we perform the same row operations on the right-hand side, and the matrix equation becomes:

$$\left[\begin{array}{cccc:cc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1/L & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/C & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hdashline -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]\left[\begin{array}{c} v_1 \\ v_3 \\ i_1 \\ i_2 \\ \hdashline v_2 \\ i_3 \end{array}\right] = \left[\begin{array}{c} I(t) \\ 0 \\ 0 \\ 0 \\ \hdashline 0 \\ 0 \end{array}\right] + \left[\begin{array}{cccc:cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & L \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]\left[\begin{array}{c} \dot{v}_1 \\ \dot{v}_3 \\ \dot{i}_1 \\ \dot{i}_2 \\ \hdashline \dot{v}_2 \\ \dot{i}_3 \end{array}\right]$$

The first four numbers in row 4 are now zero, as required, and we turn our attention to row 6.

Adding $-1 \times$ (row 1) and $-C \times$ (row 3) to row 6, we get

$$\left[\begin{array}{cccc:cc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1/L & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/C & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hdashline -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]\left[\begin{array}{c} v_1 \\ v_3 \\ i_1 \\ i_2 \\ \hdashline v_2 \\ i_3 \end{array}\right] = \left[\begin{array}{c} I(t) \\ 0 \\ 0 \\ 0 \\ \hdashline 0 \\ -I(t) \end{array}\right] + \left[\begin{array}{cccc:cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & L \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -C & 0 \end{array}\right]\left[\begin{array}{c} \dot{v}_1 \\ \dot{v}_3 \\ \dot{i}_1 \\ \dot{i}_2 \\ \hdashline \dot{v}_2 \\ \dot{i}_3 \end{array}\right] \quad (*)$$

This completes the elimination procedure. From rows 4 and 6 of the matrix equation we can write down the two state equations relating the two state variables of the network:

$$v_2 = L\dot{i}_3$$

$$i_3 = -I(t) - C\dot{v}_2$$

or, in the more usual form,

$$\dot{i}_3 = \frac{1}{L} v_2$$

$$\dot{v}_2 = -\frac{1}{C} i_3 - \frac{1}{C} I(t).$$

The general matrix form of the state equations is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{u},$$

where

- **A** is a $k \times k$ matrix;
- **x** is a column vector of the k state variables of the system;
- $\dot{\mathbf{x}}$ is a column vector of their derivatives;
- **u** is a column vector involving the independent sources.

For example, the state equations in our example can be written as

$$\begin{bmatrix} \dot{v}_2 \\ \dot{i}_3 \end{bmatrix} = \begin{bmatrix} 0 & -1/C \\ 1/L & 0 \end{bmatrix} \begin{bmatrix} v_2 \\ i_3 \end{bmatrix} + \begin{bmatrix} -(1/C)I(t) \\ 0 \end{bmatrix}$$

We now solve the state equations. The exact form of the solution depends on how the current $I(t)$ varies with time, on the initial conditions of the problem (that is, the values of v_2 and i_3 at time $t = 0$), and on the values of L and C.

In this case, the solution has the form

$$v_2(t) = A \cos \omega t + B \sin \omega t + f(t),$$

$$i_3(t) = A_1 \cos \omega t + B_1 \sin \omega t + g(t),$$

where A, A_1, B and B_1 are constants, $\omega = (LC)^{-1/2}$, and f and g are functions which depend on $I(t)$ and its derivative.

In fact,

$$A_1 = -BC\omega,$$

$$B_1 = AC\omega,$$

$$g(t) = -Cf'(t) - I(t),$$

and f is a function satisfying the differential equation

$$f''(t) + \frac{1}{LC} f(t) + \frac{1}{C} I'(t) = 0.$$

Finding the other edge variables i_1, v_3, i_2 and v_1 is now straightforward, since we can obtain them in terms of the state variables v_2 and i_3 and their derivatives. To do this, we look back at the matrix equation (*). The first four equations are

$$i_1 = I(t),$$

$$\frac{1}{L} v_3 = \dot{i}_3,$$

$$\frac{1}{C} i_2 = \dot{v}_2,$$

$$-v_1 + L\dot{i}_3 = 0,$$

and i_1, v_3, i_2 and v_1 can easily be calculated from these equations.

Problem 3.4

(a) Express the following component equations in matrix form:

$$v_1 = k, \quad i_2 = C\frac{dv_2}{dt}, \quad v_3 = L\frac{di_3}{dt}, \quad v_4 = R_4 i_4.$$

(b) Write down the **H**-matrix equation for a system which has the component equations given in part (a), and the following fundamental cycle and cutset equations:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_3 \\ v_2 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 1\cancel{0} \end{bmatrix}\begin{bmatrix} i_1 \\ i_3 \\ i_2 \\ i_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(c) Find the state equations.

The procedure carried out above for our simple example can be extended to any network made up of capacitors, inductors, resistors and independent sources, or their analogues in other types of physical system. The number of state variables, and hence state equations, needed to describe the dynamic behaviour of a system completely is related to the number of energy storage components in the system. In our simple example, we had two such components: a capacitor, which stores energy in the form of an electric field, and an inductor, which stores energy in the form of a magnetic field. We were able to write down the general form of the solution to the state equations, but for more complex networks the state equations would be solved numerically using a computer. There are various computational techniques available to tackle the problem, but a discussion of their relative advantages and drawbacks is beyond the scope of this unit.

After studying this section, you should be able to:

(a) find the component matrix for a given resistive network;

(b) combine the component matrix with the fundamental cycle and cutset matrices to form the matrix **H**;

(c) understand the use of Gaussian elimination in the solution of the matrix equation **Hx** = **y**;

(d) explain the terms *state variable* and *state equation*, formulate the matrix equation $\mathbf{Hx} = \mathbf{y} + \mathbf{K}\dot{\mathbf{x}}$ from the component, cutset and cycle equations of a given network and derive the state equations.

3.4 Computer activities

The computer activities for this section are described in the *Computer Activities Booklet*.

Further reading

Further information about electrical networks may be found in:

C. A. Desoer and Ernest S. Kuh, *Basic Circuit Theory*, McGraw-Hill, 1969;

S. W. Director, *Circuit Theory: A Computational Approach*, John Wiley and Sons, 1975;

W. A. Blackwell, *Mathematical Modelling of Physical Networks*, MacMillan, 1968.

Further information about physical networks in general may be found in:

P. H. Roe, *Networks and Systems*, Addison-Wesley, 1966.

A useful account of matrix theory, illustrated with many examples, is given in:

F. Ayres, Schaum's *Outline of Theory and Problems of Matrices*, McGraw-Hill, 1962.

Acknowledgement

Grateful acknowledgement is made to the Science Museum for permission to reproduce the picture of Kirchhoff on page 8.

Exercises

Section 1

Components, and through and across variables

1.1 Classify each of the following as a *through* variable or an *across* variable:

(a) the *fluid flow rate* for a hydraulic valve;

(b) the *force* acting on a mass;

(c) the *derivative of the voltage* for a capacitor.

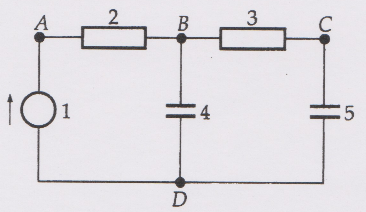
1.2 Identify the through and across variables, and write down the component equation, for each of the following components:

(a) an electrical inductor;

(b) a spring;

(c) a transformer (considered as a 3-terminal component).

1.3 Consider the following system:

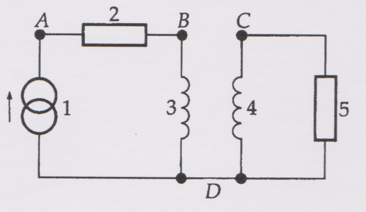


(a) Draw an oriented graph representation of the system.

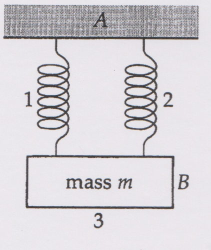
(b) Apply Kirchhoff's voltage law to each cycle of the oriented graph.

(c) Apply Kirchhoff's current law to each vertex of the oriented graph.

1.4 Repeat Exercise 1.3 for the following system:



1.5 Repeat Exercise 1.3 for the following system:



Graphical representation

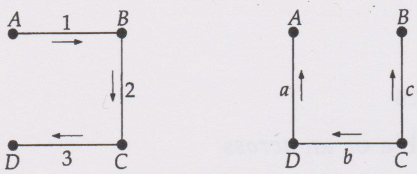
1.6 Decide whether each of the following statements is TRUE or FALSE.

(a) In an oriented graph, a 6-terminal component can always be represented by 6 vertices and 5 edges.

(b) In an oriented graph representation of an electrical network, the arrow next to an edge always points in the same direction as the actual direction of the current flow in the corresponding component.

(c) In an oriented graph, a 5-terminal component can be represented in exactly 5 different ways (ignoring edge directions).

1.7 The following graphs can both be used to represent the same 4-terminal component:

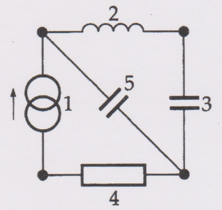


(a) Express each of the voltage variables v_a, v_b and v_c in terms of v_1, v_2 and v_3.

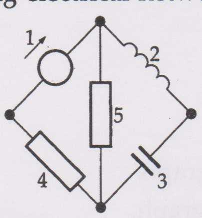
(b) Express each of the current variables i_a, i_b and i_c in terms of i_1, i_2 and i_3.

1.8 Draw an oriented graph representation of a 5-terminal component. How many component equations are associated with such a component?

1.9 Draw an oriented graph which represents the following electrical network:



1.10 Consider the following electrical network.



(a) Construct the dual network.

(b) Write down the component equations of the dual network, given that the component equations of the original network are

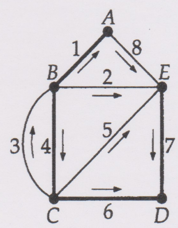
1 2 5 4 3

$$v_1 = 3, \quad v_2 = \cancel{10i_2}\ 4\frac{di_2}{dt}, \quad v_5 = 5i_5, \quad v_4 = \cancel{4\frac{di_4}{dt}}\ 10i_4, \quad i_3 = 6\frac{dv_3}{dt}.$$

Section 2

Fundamental cycles and cutsets

2.1 Consider the following oriented graph, in which a spanning tree is indicated by thick edges:



(a) List the fundamental cycles and write down the corresponding voltage law equations.

(b) List the fundamental cutsets and write down the corresponding current law equations.

(c) Find the fundamental cycle matrix $\mathbf{C}_f$.

(d) Find the fundamental cutset matrix $\mathbf{D}_f$.

2.2 An oriented graph has 8 vertices and 12 edges. Write down:

(a) the number of edges in a spanning tree;

(b) the number of edges in each co-tree;

(c) the number of fundamental cycles;

(d) the number of fundamental cutsets.

Obtaining the fundamental cycle and cutset equations

2.3 Consider the oriented graph of Exercise 2.1, with vertex A as reference vertex.

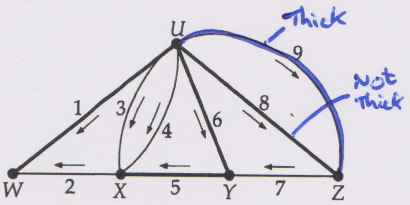
(a) Write down the reduced incidence matrix, and partition it into a tree part $\mathbf{B}_t$ and a co-tree part $\mathbf{B}_c$.

(b) Use the method described in Section 2.5 to find the inverse of the matrix $\mathbf{B}_t$.

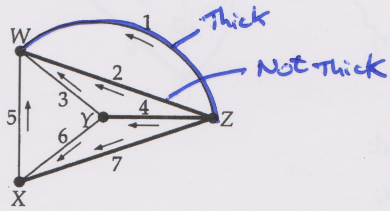
(c) Use the matrices $\mathbf{B}_t^{-1}$ and $\mathbf{B}_c$ to find the fundamental cycle matrix $\mathbf{C}_f$ and the fundamental cutset matrix $\mathbf{D}_f$.

(d) Calculate the matrix product $\mathbf{C}_f\mathbf{D}_f^T$.

2.4 For each of the following oriented graphs, write down the fundamental cycle equations and the fundamental cutset equations, using the spanning tree indicated by *thick* edges.



(a)



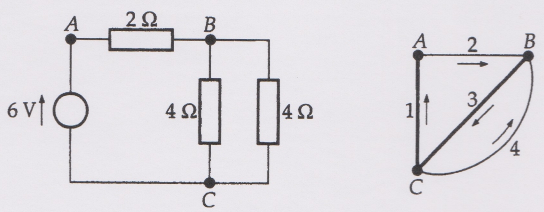
(b)

Hint These oriented graphs are the duals of those on pages 40 and 41. Use the results stated in Solution 2.10.

Section 3

Formulation of the matrix equations

3.1 The following diagram shows an electrical network and an associated oriented graph:



The resistances are given in ohms (Ω).

(a) Using the spanning tree shown, find the component equations and the fundamental cycle and cutset equations.

(b) Solve the equations to find all the currents and voltages in the network.

3.2

(a) Express the following component equations in matrix form:

$$v_1 = R_1 i_1,\ \ v_2 = 3,\ \ v_3 = R_3 i_3,\ \ v_4 = R_4 i_4.$$

(b) Write down the **H**-matrix equation for a system which has the component equations given in part (a), and the following fundamental cycle and cutset equations:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_3 \\ v_2 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} i_1 \\ i_3 \\ i_2 \\ i_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

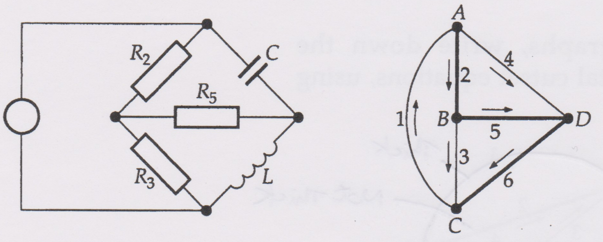
Solving the matrix equations

3.3 Use Gaussian elimination to solve the following matrix equation:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \\ -13 \end{bmatrix}.$$

State equations

3.4 The following diagram shows an electrical network and an associated oriented graph:



Using the spanning tree shown, find:

(a) the fundamental cycle matrix;

(b) the fundamental cutset matrix;

(c) the component equations;

(d) the **H**-matrix equation;

(e) one of the state equations.

Solutions to the exercises

1.1

(a) through variable; (b) through variable; (c) across variable.

1.2

(a) The through variable is the current i and the across variable is the voltage v. The component equation is

$$v = L\frac{di}{dt},$$

where L is the inductance.

(b) The through variable is the applied force f, and the across variable is the extension x. The component equation is

$$f = kx,$$

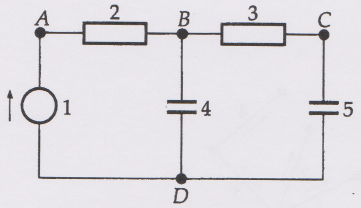
where k is a constant.

(c) Since the transformer has three terminals, it has two components. The through variables are the currents i_1 and i_2 flowing in the input and output windings, and the across variables are the voltages v_1 and v_2 across these windings. The component equations are

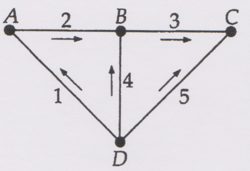
$$v_1 = (n_1/n_2)v_2 \quad \text{and} \quad i_1 = -(n_2/n_1)i_2,$$

where n_1 and n_2 are the numbers of turns in the two windings.

1.3

(a) 

system

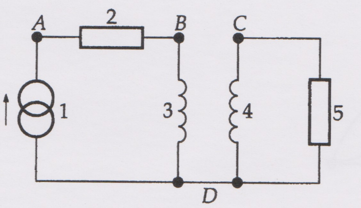


oriented graph

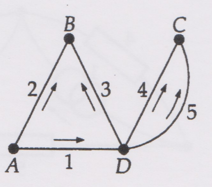
(b) cycle $ABDA$ $\quad v_2 - v_4 + v_1 = 0;$
cycle $BCDB$ $\quad v_3 - v_5 + v_4 = 0;$
cycle $ABCDA$ $\quad v_2 + v_3 - v_5 + v_1 = 0.$

(c) vertex A $\quad -i_1 + i_2 = 0;$
vertex B $\quad -i_2 - i_4 + i_3 = 0;$
vertex C $\quad -i_3 - i_5 = 0;$
vertex D $\quad i_1 + i_4 + i_5 = 0.$

1.4

(a) 

system



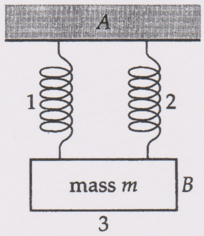
oriented graph

(b) cycle $ABDA$ $\quad v_2 - v_3 - v_1 = 0;$
cycle CDC $\quad -v_5 + v_4 = 0.$

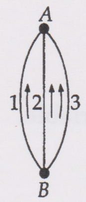
(c) vertex A $\quad i_1 + i_2 = 0;$
vertex B $\quad -i_2 - i_3 = 0;$
vertex C $\quad -i_4 - i_5 = 0;$
vertex D $\quad -i_1 + i_3 + i_4 + i_5 = 0.$

1.5

(a)



system



oriented graph

(b) cycle ABA (left) $-v_2 + v_1 = 0$;
cycle ABA (right) $-v_3 + v_2 = 0$;
cycle ABA (outer) $-v_3 + v_1 = 0$.

(c) vertex A $-i_1 - i_2 - i_3 = 0$;
vertex B $i_1 + i_2 + i_3 = 0$.

1.6

(a) TRUE; (b) FALSE; (c) FALSE.

1.7

(a) Clearly $v_b = v_3$ and $v_c = -v_2$.

The potential difference across the terminals D and A is

$$v_a = -v_1 - v_2 - v_3.$$

(b) Applying Kirchhoff's current law to the vertices A, B and C, we have

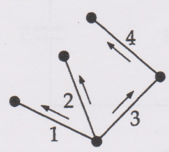
$$i_a = -i_1, \quad i_c = i_1 - i_2, \quad i_b + i_c = i_3 - i_2,$$

giving

$$i_a = -i_1, \quad i_b = i_3 - i_1, \quad i_c = i_1 - i_2.$$

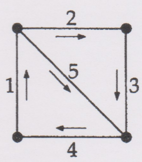
1.8

There are several possibilities — for example:



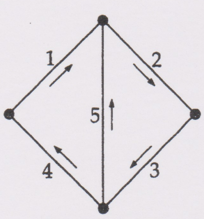
All oriented graph representations are trees with four edges, so there are four component equations associated with a 5-terminal component.

1.9

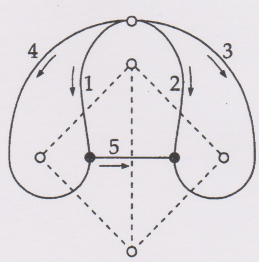


1.10

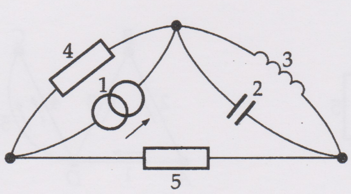
(a)



oriented graph



dual graph



dual network

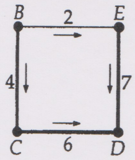
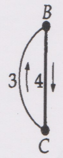
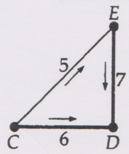
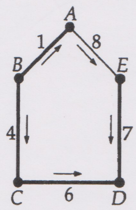
Notice how the cycles/cutsets in the dual oriented graph correspond to the cutsets/cycles in the original oriented graph; for example, look at the edges marked 1, 5, 4 and those marked 1, 5, 2 in each oriented graph.

(b) $i_1 = 3$, $i_2 = \cancel{10v_2}\ 4\frac{dv_2}{dt}$, $i_4 = \cancel{5v_3}\ 10v_4$, $i_5 = \cancel{4\frac{dv_4}{dt}}\ 5v_5$, $v_3 = 6\frac{di_3}{dt}$.

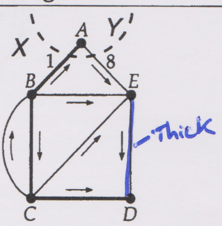
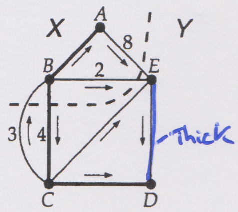
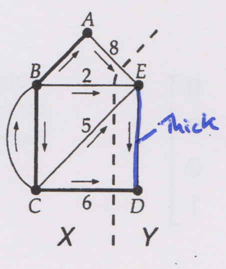
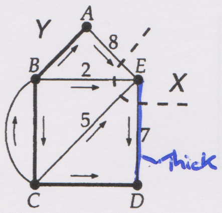
1 2 4 5 3

2.1

(a)

chord	fundamental cycle	diagram	fundamental cycle equation
BE	*BEDCB*		$v_2 + v_7 - v_6 - v_4 = 0$
CB	*CBC*		$v_3 + v_4 = 0$
CE	*CEDC*		$v_5 + v_7 - v_6 = 0$
AE	*AEDCBA*		$v_8 + v_7 - v_6 - v_4 + v_1 = 0$

(b)

branch	fundamental cutset	diagram	fundamental cutset equation
BA	{*BA*, *AE*}		$i_1 - i_8 = 0$
BC	{*BC*, *CB*, *BE*, *AE*}		$i_4 - i_3 + i_2 + i_8 = 0$
CD	{*CD*, *CE*, *BE*, *AE*}		$i_6 + i_5 + i_2 + i_8 = 0$
ED	{*ED*, *CE*, *BE*, *AE*}		$i_7 - i_5 - i_2 - i_8 = 0$

(c)

$$\mathbf{C}_f = \begin{array}{cc} & \begin{array}{cccccccc} BA & BC & CD & ED & BE & CB & CE & AE \end{array} \\ \begin{array}{c} BE \\ CB \\ CE \\ AE \end{array} & \begin{bmatrix} 0 & -1 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(d)

$$\mathbf{D}_f = \begin{array}{cc} & \begin{array}{cccccccc} BA & BC & CD & ED & BE & CB & CE & AE \end{array} \\ \begin{array}{c} BA \\ BC \\ CD \\ DE \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 & -1 \end{bmatrix} \end{array}$$

2.2

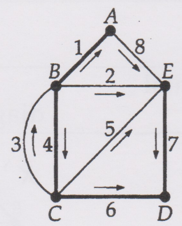
(a) 7; (b) 5; (c) 5 (one for each chord); (d) 7 (one for each branch).

2.3

(a) The reduced incidence matrix is

$$\begin{array}{cc} & \begin{array}{cccc|cccc} BA & BC & CD & ED & BE & CB & CE & Æ \end{array} \\ \begin{array}{c} B \\ C \\ D \\ E \end{array} & \left[\begin{array}{cccc:cccc} 1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 & -1 \end{array}\right] \\ & \quad \mathbf{B}_t \qquad\qquad\qquad \mathbf{B}_c \end{array}$$

(b) Tracing a path in the tree from each vertex to the reference vertex, we find that the edges ED and BA are traversed in the direction of their arrows, and CD and BC are traversed in the opposite direction. Thus the matrix $\mathbf{B}_t^{-1}$ is



$$\begin{array}{cc} & \begin{array}{cccc} B & C & D & E \end{array} \\ \begin{array}{c} BA \\ BC \\ CD \\ ED \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(c) The matrix product $\mathbf{B}_t^{-1}\mathbf{B}_c$ is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ -1 & 0 & -1 & -1 \end{bmatrix}$$

The fundamental cycle matrix is

$$\mathbf{C}_f = [-(\mathbf{B}_t^{-1}\mathbf{B}_c)^T \mid \mathbf{I}_4] = \left[\begin{array}{cccc:cccc} 0 & -1 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

Note that these answers agree with those given in the solution to Exercise 2.1.

The fundamental cutset matrix is

$$\mathbf{D}_f = \left[\mathbf{I}_4 \mid \mathbf{B}_t^{-1}\mathbf{B}_c\right] = \left[\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 & -1 \end{array}\right]$$

(d) $\mathbf{C}_f\mathbf{D}_f^T = \mathbf{0}$.

2.4

Using the results on pages 40 and 41 and the results about dual oriented graphs, we can immediately write down the equations as follows.

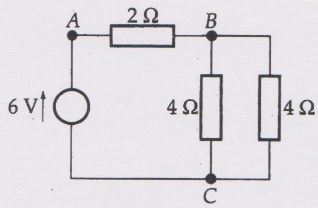
For the fundamental cycle equations, we replace i by v in the fundamental cutset equations and obtain the following:

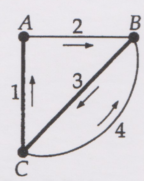
graph (a)	graph (b)
$v_3 - v_5 - v_6 = 0$	$v_2 - v_1 = 0$
$v_2 + v_6 + v_5 - v_1 = 0$	$v_3 + v_4 - v_1 = 0$
$v_7 - v_6 + v_9 = 0$	$v_6 + v_4 - v_7 = 0$
$v_8 - v_9 = 0$	$v_5 + v_7 - v_1 = 0$
$v_4 - v_6 - v_5 = 0$	

For the fundamental cutset equations, we replace v by i in the fundamental cycle equations and obtain the following:

graph (a)	graph (b)
$i_3 + i_6 + i_7 + i_4 - i_2 = 0$	$i_4 - i_6 - i_3 = 0$
$i_3 + i_5 + i_4 - i_2 = 0$	$i_6 + i_7 - i_5 = 0$
$i_9 + i_8 - i_7 = 0$	$i_2 + i_3 + i_5 + i_1 = 0$
$i_2 + i_1 = 0$	

3.1





(a) The component equations are:

component 1 (voltage source) $\quad v_1 = -6 \quad$ (1)
component 2 (2 Ω resistor) $\quad v_2 = 2i_2 \quad$ (2)
component 3 (4 Ω resistor) $\quad v_3 = 4i_3 \quad$ (3)
component 4 (4 Ω resistor) $\quad v_4 = 4i_4 \quad$ (4)

Note that the component equation for the voltage source has a minus sign in it. This is because the reference arrow for the edge corresponding to the voltage source defines a positive voltage as one which is in the *opposite* direction to that of the voltage source in the network diagram.

The fundamental cycle equations are:

chord	cycle	cycle equation	
AB	$ABCA$	$v_2 + v_3 + v_1 = 0$	(5)
BC	BCB	$v_3 + v_4 = 0$	(6)

The fundamental cutset equations are:

chord	cutset	cutset equation	
CA	$\{CA,\ AB\}$	$i_1 - i_2 = 0$	(7)
BC	$\{BC,\ AB,\ CB\}$	$i_3 - i_2 - i_4 = 0$	(8)

(b) We have eight equations involving eight variables. We solve these equations as follows. Substituting equations (1), (2), (3) and (4) into equations (5) and (6), we get

$$2i_2 + 4i_3 - 6 = 0, \quad \text{so} \quad i_2 = 3 - 2i_3 \qquad (= i_1 \text{ from equation (7)})$$

and

$$4i_3 + 4i_4 = 0, \quad \text{so} \quad i_4 = -i_3.$$

Substituting into equation (8), we get

$$i_3 - 3 + 2i_3 + i_3 = 0, \quad \text{so} \quad i_3 = \frac{3}{4}\text{ amps.}$$

Hence

$$i_1 = i_2 = \frac{3}{2}\text{ amps};\quad i_3 = \frac{3}{4}\text{ amps};\quad i_4 = -\frac{3}{4}\text{ amps};$$

from equations (1), (2), (3) and (4), we now get

$$v_1 = -6\text{ volts};\quad v_2 = 3\text{ volts};\quad v_3 = 3\text{ volts};\quad v_4 = -3\text{ volts}.$$

3.2

(a) The component equation matrix is

$$\left[\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & -R_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -R_3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -R_4 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} v_1 \\ v_3 \\ v_4 \\ v_2 \\ i_1 \\ i_3 \\ i_4 \\ i_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \end{bmatrix}$$

(b) Before forming the **H**-matrix we first rewrite the matrix of part (a) so that the voltages and currents in the column vector appear in the same order as that of the fundamental cycle and cutset matrices. The component equation matrix thus becomes:

$$\left[\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & -R_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -R_3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -R_4 \end{array}\right]$$

The **H**-matrix equation **Hx** = **y** is therefore

$$\left[\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & -R_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -R_3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -R_4 \\ \hdashline 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right] \begin{bmatrix} v_1 \\ v_3 \\ v_2 \\ v_4 \\ \hdashline i_1 \\ i_3 \\ \hdashline i_4 \\ i_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \\ \hdashline 0 \\ 0 \\ \hdashline 0 \\ 0 \end{bmatrix}$$

3.3

We are given the matrix equation

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \\ -13 \end{bmatrix}$$

We first modify the first column by using the first row. To do this, we subtract the first row from the second and third rows to give

row 2: 0 2 0 −1
row 3: 0 0 1 0

The matrix equation can now be written as:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -3 \\ -13 \end{bmatrix}$$

The second column is already in the required form, so we turn our attention to the third column. To make the bottom element (3) zero, we must multiply the third row by 3 and subtract it from the fourth row. The matrix equation then becomes:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -3 \\ -4 \end{bmatrix}$$

We now find the values of the variables by back-substitution.

From the fourth equation, $x_4 = 4$.

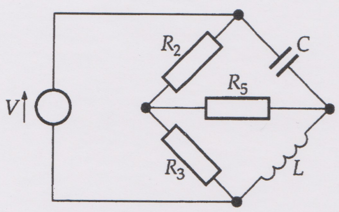
From the third equation, $x_3 = -3$.

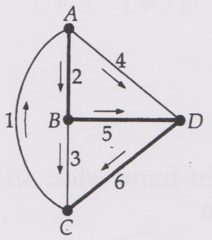
The second equation is $2x_2 - x_4 = 0$, giving $x_2 = 2$.

The first equation is $x_1 + x_4 = 3$, giving $x_1 = -1$.

So the solution of the matrix equation is $x_1 = -1$, $x_2 = 2$, $x_3 = -3$ and $x_4 = 4$.

3.4





(a)

chord	cycle	cycle equation
CA	$CABDC$	$v_1 + v_2 + v_5 + v_6 = 0$
BC	$BCDB$	$v_3 - v_6 - v_5 = 0$
AD	$ADBA$	$v_4 - v_5 - v_2 = 0$

The fundamental cycle matrix is

$$\begin{array}{c} \\ \\ CA \\ BC \\ AD \end{array} \begin{array}{c} \begin{array}{cccccc} 2 & 5 & 6 & 1 & 3 & 4 \\ AB & BD & DC & CA & BC & AD \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(b)

branch	cutset	cutset equation
AB	$\{AB,\ AD,\ CA\}$	$-i_1 + i_2 + i_4 = 0$
BD	$\{BD,\ AD,\ CA,\ BC\}$	$-i_1 + i_3 + i_4 + i_5 = 0$
DC	$\{DC,\ CA,\ BC\}$	$-i_1 + i_3 + i_6 = 0$

The fundamental cutset matrix is

$$\begin{array}{c} \\ \\ AB \\ BD \\ DC \end{array} \begin{array}{c} \begin{array}{cccccc} 2 & 5 & 6 & 1 & 3 & 4 \\ AB & BD & DC & CA & BC & AD \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{bmatrix} \end{array}$$

(c) The component equations are:

voltage source $\quad v_1 = -V$

resistors $\quad v_2 = R_2 i_2,\quad v_3 = R_3 i_3,\quad v_5 = R_5 i_5$

capacitor $\quad i_4 = C\dfrac{dv_4}{dt}$

inductor $\quad v_6 = L\dfrac{di_6}{dt}$

(d) The **H**-matrix equation is $\mathbf{Hx} = \mathbf{y} + \mathbf{K\dot{x}}$, which can be written as follows. (Note that the variables v_4 and i_6, whose derivatives occur in the component equations, appear last.)

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -R_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -R_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -R_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/C & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/L & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_5 \\ v_6 \\ i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ v_4 \\ i_6 \end{bmatrix} = \begin{bmatrix} -V \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \\ \dot{v}_5 \\ \dot{v}_6 \\ \dot{i}_1 \\ \dot{i}_2 \\ \dot{i}_3 \\ \dot{i}_4 \\ \dot{i}_5 \\ \dot{v}_4 \\ \dot{i}_6 \end{bmatrix}$$

(e) We can eliminate the (–1)s in row 9 by taking row 9 + row 7 – row 1 – ($L \times$ row 6), giving

0 0 0 0 0 0 0 0 0 0 1 0

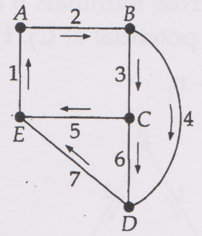
Applying these row operations to the right-hand side of the **H**-matrix equation, we obtain the state equation

$$v_4 = V - L\dot{i}_6.$$

Solutions to the problems

Solution 1.1

There are several possible oriented graphs, depending on how we label the components and which direction we assign to each edge. One possibility is shown in the margin.



Solution 1.2

Differentiating the given equation, we obtain

$$\frac{df}{dt} = k\frac{dx}{dt} = ku$$

or

$$u = \frac{1}{k}\frac{df}{dt}.$$

In this case, f is the through variable and u is the across variable. This equation is of the same form as the component equation for an inductor,

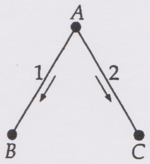
$$v = L\frac{di}{dt}$$

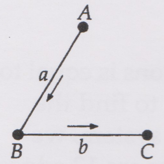
so the spring is analogous to an *inductor,* and the constant $1/k$ (the reciprocal of the spring stiffness) corresponds to the inductance L. The analogous quantities for the two components are shown in the following table.

	mechanical	electrical
component	spring	inductor
through variable	f	i
across variable	u	v
component equation	$u = \frac{1}{k}\frac{df}{dt}$	$v = L\frac{di}{dt}$

Solution 1.3

(a)





Applying Kirchhoff's current law to the vertex A, we obtain

current flowing into $A = i_a$.

Applying Kirchhoff's current law to the vertex A for the original graphical representation, we obtain

current flowing into $A = i_1 + i_2$.

Hence

$i_a = i_1 + i_2$.

Applying Kirchhoff's current law to the vertex C, we obtain

current flowing into $C = -i_b$.

Applying Kirchhoff's current law to the vertex C for the graphical representation in the text, we obtain

current flowing into $C = -i_2$.

Hence

$i_b = i_2$.

The potential difference (or voltage) across terminals A and B is just v_a. (The direction of the arrow indicates that this is considered positive if A is at a higher potential than B.) Looking back at the graphical representation in the text, we see that the potential difference across terminals A and B is v_1. Hence

$v_a = v_1$.

The potential difference across terminals B and C is v_b. Looking at the graphical representation in the text, we see that the potential difference across terminals B and C is $v_2 - v_1$ (that is, the potential at B minus the potential at C). Hence

$$v_b = v_2 - v_1.$$

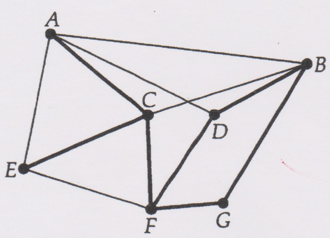
(b)

Using the same method as in part (a), we obtain:

$$i_a = i_1 + i_2, \quad i_b = -i_1, \quad v_a = v_2, \quad v_b = v_2 - v_1.$$

Solution 1.4

(a) To find a graphical representation of this system, we replace each 2-terminal component by an edge, each 3-terminal component by a tree with two edges, and the 4-terminal component by a tree with three edges. Since each tree can be chosen in several different ways, there are many possible ways of representing this system. One possible way is the following. The tree edges are shown by thick lines.

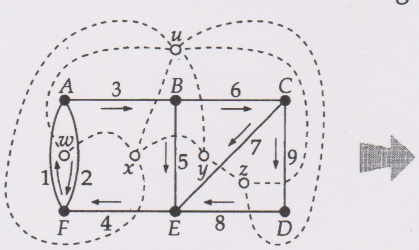


component		edges
ACE	(3-terminal)	*AC*, *CE*
BDG	(3-terminal)	*BD*, *BG*
CDFG	(4-terminal)	*CF*, *DF*, *FG*

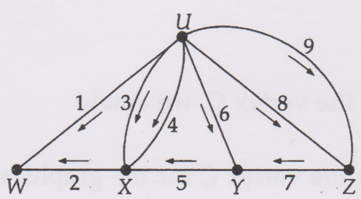
(b) For any component, the number of component equations is equal to the number of edges in its graphical representation. So, to find the number of component equations associated with the system, we simply count the number of edges in the graph. There are 12 edges, so there are 12 component equations.

Solution 1.5

(a) We construct the dual oriented graph as follows:



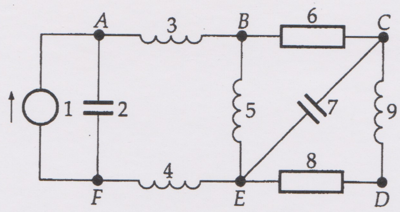
oriented graph



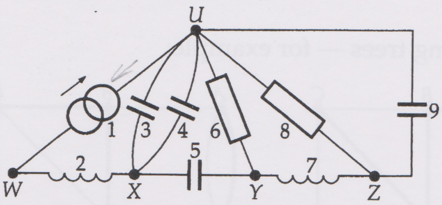
dual oriented graph

We obtain the dual network shown below by replacing

component 1 by a current source;
components 2 and 7 by inductors;
components 3, 4, 5 and 9 by capacitors.

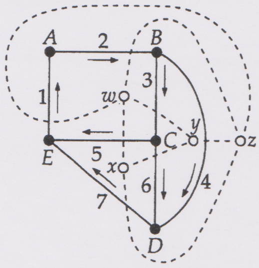


network

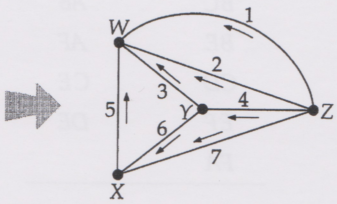


dual network

(b) We construct the dual oriented graph as follows:



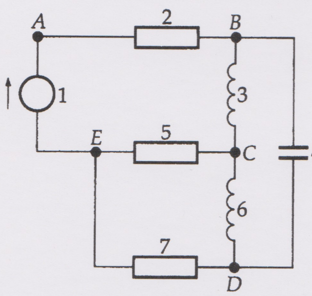
oriented graph



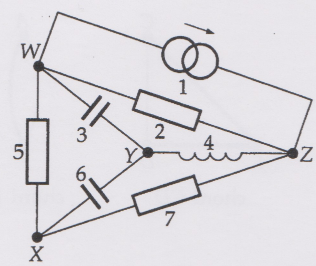
dual oriented graph

We obtain the dual network shown below by replacing

component 1 by a current source;
components 3 and 6 by capacitors;
component 4 by an inductor.



network

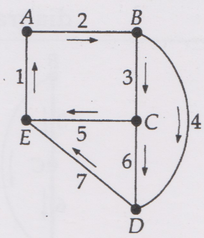


dual network

Solution 2.1

(a)

cycle	voltage equation
$ABCEA$	$v_2 + v_3 + v_5 + v_1 = 0$ (1)
$BDCB$	$v_4 - v_6 - v_3 = 0$ (2)
$CDEC$	$v_6 + v_7 - v_5 = 0$ (3)
$ABDCEA$	$v_2 + v_4 - v_6 + v_5 + v_1 = 0$ (4)
$ABCDEA$	$v_2 + v_3 + v_6 + v_7 + v_1 = 0$ (5)
$BDECB$	$v_4 + v_7 - v_5 - v_3 = 0$ (6)
$ABDEA$	$v_2 + v_4 + v_7 + v_1 = 0$ (7)



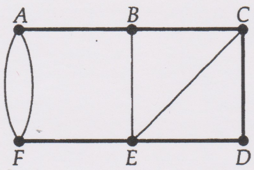
(b) The maximum number of linearly independent equations is 3. For example, equations (1), (2) and (3) are linearly independent, since each contains at least one term not included in either of the other two, and so none can depend on the other two. However, the remaining equations (4)–(7) all depend on the first three, as follows:

equation (4) = equation (1) + equation (2)
equation (5) = equation (1) + equation (3)
equation (6) = equation (2) + equation (3)
equation (7) = equation (1) + equation (2) + equation (3).

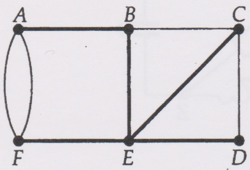
It follows that, if we wish to study the network, we need choose only equations (1), (2) and (3) (or any other 3 linearly independent equations), since the remaining equations give us no futher information.

Solution 2.2

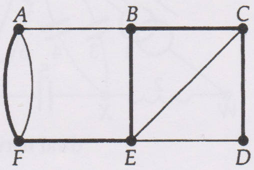
(a) There are many possible spanning trees — for example:



branches	chords
AB	*AF*
BC	*BE*
CD	*CE*
DE	*FA*
EF	



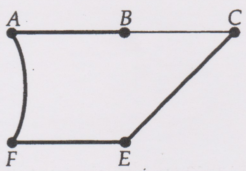
branches	chords
AB	*AF*
BE	*BC*
CE	*CD*
DE	*FA*
EF	



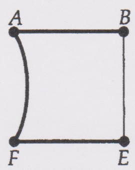
branches	chords
BC	*AB*
BE	*AF*
CD	*CE*
EF	*DE*
FA	

(b) We know that every tree with n vertices has $n-1$ edges. Thus there are $n-1$ branches, and so there are $m-(n-1)=m-n+1$ chords associated with each spanning tree.

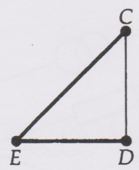
(c) If we add a chord to a spanning tree, we obtain a single cycle. For example, if we add each chord separately to the spanning tree in our example, we get the following cycles:



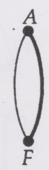
chord *BC*



chord *BE*



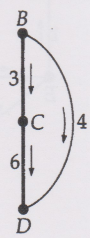
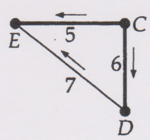
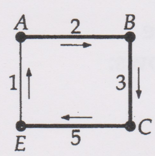
chord *CD*



chord *FA*

Solution 2.3

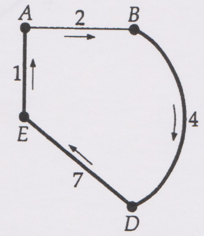
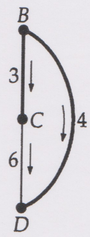
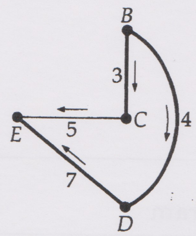
(a)

chord	fundamental cycle	diagram	fundamental cycle equation
BD	*BDCB*		$v_4 - v_6 - v_3 = 0$
DE	*CDEC*		$v_6 + v_7 - v_5 = 0$
EA	*ABCEA*		$v_2 + v_3 + v_5 + v_1 = 0$

The cycle *ABCDEA* contains the chords *DE* and *EA*. We obtain the voltage law equation for this cycle by combining the cycle equations for these two chords, giving

$$v_2 + v_3 + v_6 + v_7 + v_1 = 0.$$

(b)

chord	fundamental cycle	diagram	fundamental cycle equation
AB	*ABDEA*		$v_2 + v_4 + v_7 + v_1 = 0$
CD	*BCDB*		$v_3 + v_6 - v_4 = 0$
CE	*BCEDB*		$v_3 + v_5 - v_7 - v_4 = 0$

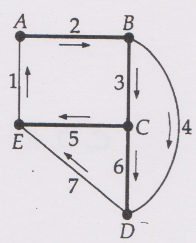
The cycle *ABCDEA* contains the chords *AB* and *CD*. We obtain the voltage law equation for this cycle by combining the cycle equations for these two chords, giving

$$v_2 + v_6 + v_3 + v_7 + v_1 = 0.$$

In this case we wish to eliminate v_4, so we combine the two equations by *addition*.

Solution 2.4

(a) The chords are *BD*, *DE* and *EA*, and the fundamental cycles are *BDCB*, *CDEC* and *ABCEA*. The fundamental cycle matrix is therefore



$$\begin{array}{c} \\ BD \\ DE \\ EA \end{array}\begin{array}{c} \begin{array}{ccccccc} AB & BC & CD & CE & BD & DE & EA \end{array} \\ \begin{bmatrix} 0 & -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

and the fundamental cycle equations are given by

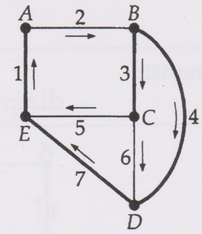
$$\begin{bmatrix} 0 & -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} v_2 \\ v_3 \\ v_6 \\ v_5 \\ v_4 \\ v_7 \\ v_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Multiplying the two matrices on the left, we obtain the equation

$$\begin{bmatrix} -v_3 - v_6 + v_4 \\ v_6 - v_5 + v_7 \\ v_2 + v_3 + v_5 + v_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which is equivalent to the three equations on page 74.

(b) The chords are AB, CD and CE, and the fundamental cycles are $ABDEA$, $BCDB$ and $BCEDB$. The fundamental cycle matrix is therefore



$$
\begin{array}{c}
\\ AB \\ CD \\ CE
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} BC & BD & DE & EA & AB & CD & CE \end{array} \\
\begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 1 & 0 \\
1 & -1 & -1 & 0 & 0 & 0 & 1
\end{bmatrix}
\end{array}
$$

and the fundamental cycle equations are given by

$$
\begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 1 & 0 \\
1 & -1 & -1 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} v_3 \\ v_4 \\ v_7 \\ v_1 \\ v_2 \\ v_6 \\ v_5 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

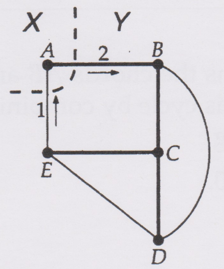
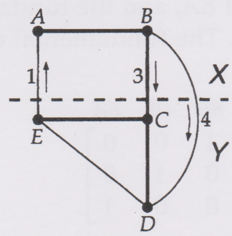
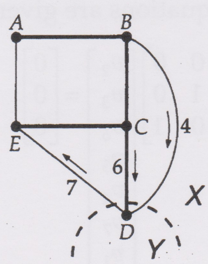
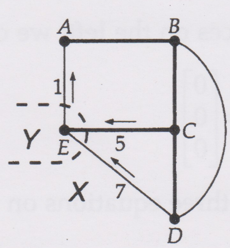
Multiplying the two matrices on the left, we obtain the equation

$$
\begin{bmatrix} v_4 + v_7 + v_1 + v_2 \\ v_3 - v_4 + v_6 \\ v_3 - v_4 - v_7 + v_5 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

which is equivalent to the three equations on page 75.

Solution 2.5

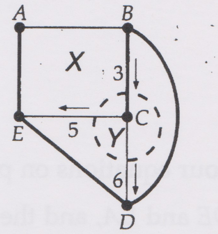
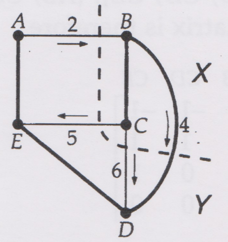
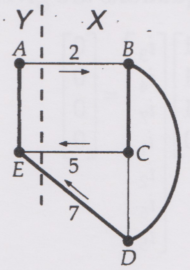
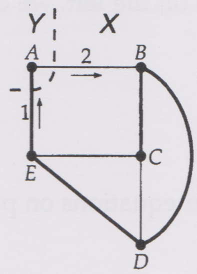
(a)

branch	vertices in X	vertices in Y	fundamental cutset	diagram	fundamental cutset equation
AB	A	B, C, D, E	$\{AB, EA\}$		$i_2 - i_1 = 0$
BC	A, B	C, D, E	$\{BC, BD, EA\}$		$i_3 + i_4 - i_1 = 0$
CD	A, B, C, E	D	$\{BD, CD, DE\}$		$i_4 + i_6 - i_7 = 0$
CE	A, B, C, D	E	$\{CE, DE, EA\}$		$i_5 + i_7 - i_1 = 0$

The cutset $\{AB, BC, BD\}$ contains the branches AB and BC. We obtain the current law equation for this cutset by combining the cutset equations for these two branches, giving

$$i_3 + i_4 - i_2 = 0.$$

(b)

branch	vertices in X	vertices in Y	fundamental cutset	diagram	fundamental cutset equation
BC	A, B, D, E	C	$\{BC, CD, CE\}$		$i_3 - i_5 - i_6 = 0$
BD	B, C	A, D, E	$\{AB, BD, CD, CE\}$		$i_4 + i_5 + i_6 - i_2 = 0$
DE	B, C, D	A, E	$\{AB, CE, DE\}$		$i_5 + i_7 - i_2 = 0$
EA	B, C, D, E	A	$\{AB, EA\}$		$i_1 - i_2 = 0$

The cutset $\{AB, BC, BD\}$ contains the branches BC and BD. We obtain the current law equation for this cutset by combining the cutset equations for these two branches, giving

$$i_3 + i_4 - i_2 = 0.$$

Solution 2.6

(a) The branches are AB, BC, CD and CE, and the fundamental cutsets are $\{AB, EA\}$, $\{BC, BD, EA\}$, $\{BD, CD, DE\}$ and $\{CE, DE, EA\}$. The fundamental cutset matrix is therefore

$$\begin{array}{c} \\ AB \\ BC \\ CD \\ CE \end{array} \begin{array}{c} \begin{array}{ccccccc} AB & BC & CD & CE & BD & DE & EA \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 \end{bmatrix} \end{array}$$

and the fundamental cutset equations are given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} i_2 \\ i_3 \\ i_6 \\ i_5 \\ i_4 \\ i_7 \\ i_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Multiplying the two matrices on the left, we obtain the equation

$$\begin{bmatrix} i_2 - i_1 \\ i_3 + i_4 - i_1 \\ i_6 + i_4 - i_7 \\ i_5 + i_7 - i_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which is equivalent to the four equations on page 76.

(b) The branches are BC, BD, DE and EA, and the fundamental cutsets are $\{BC, CD, CE\}$, $\{AB, BD, CD, CE\}$, $\{AB, CE, DE\}$ and $\{AB, EA\}$. The fundamental cutset matrix is therefore

$$\begin{array}{c} \\ BC \\ BD \\ DE \\ EA \end{array} \begin{array}{c} \begin{array}{ccccccc} BC & BD & DE & EA & AB & CD & CE \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix} \end{array}$$

and the fundamental cutset equations are given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_3 \\ i_4 \\ i_7 \\ i_1 \\ i_2 \\ i_6 \\ i_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Multiplying the two matrices on the left, we obtain the equation

$$\begin{bmatrix} i_3 - i_6 - i_5 \\ i_4 - i_2 + i_6 + i_5 \\ i_7 - i_2 + i_5 \\ i_1 - i_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which is equivalent to the four equations on page 77.

Solution 2.7

(a)

$$\mathbf{B} = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccccc} AB & BC & BD & CD & CE & DE & EA \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 \end{bmatrix} \end{array}$$

(b)

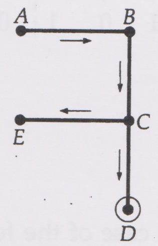
$$\mathbf{B}_0 = \begin{array}{c} \\ A \\ B \\ C \\ E \end{array} \begin{array}{c} \begin{array}{ccccccc} AB & BC & BD & CD & CE & DE & EA \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 \end{bmatrix} \end{array}$$

(c)

$$\mathbf{B}_t = \begin{array}{c} \\ A \\ B \\ C \\ E \end{array}\!\!\begin{array}{c} \begin{array}{cccc} AB & BC & CD & CE \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \end{array} \qquad \mathbf{B}_c = \begin{array}{c} \\ A \\ B \\ C \\ E \end{array}\!\!\begin{array}{c} \begin{array}{ccc} BD & DE & EA \end{array} \\ \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} \end{array}$$

Solution 2.8

We first draw the spanning tree and indicate D as the chosen reference vertex.



Tracing the path from each vertex to the reference vertex, we obtain the matrix

$$\mathbf{B}_t^{-1} = \begin{array}{c} \\ AB \\ BC \\ CD \\ CE \end{array}\!\!\begin{array}{c} \begin{array}{cccc} A & B & C & E \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \end{array}$$

We have

$$\mathbf{B}_t^{-1}\mathbf{B}_t = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{I}_4,$$

$$\mathbf{B}_t\mathbf{B}_t^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{I}_4,$$

as required.

Solution 2.9

(a)

$$\mathbf{B}_t^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad \mathbf{B}_c = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Hence

$$\mathbf{B}_t^{-1}\mathbf{B}_c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

and so

$$\mathbf{D}_f = \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right]$$

which agrees with the answer given in the solution to Problem 2.6.

Also

$$(\mathbf{B}_t^{-1}\mathbf{B}_c)^T = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & -1 & 0 & -1 \end{bmatrix}$$

and so

$$\mathbf{C}_f = \left[\begin{array}{cccc|ccc} 0 & -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

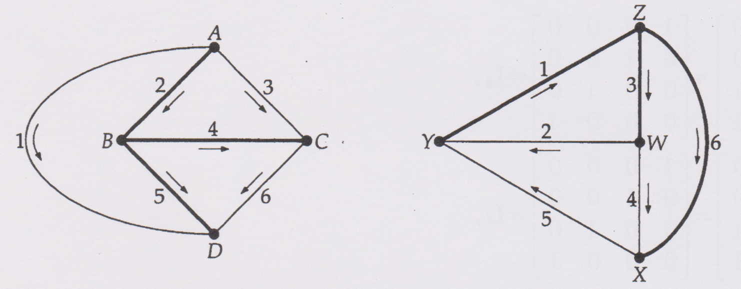
which agrees with the answer given in the solution to Problem 2.4.

(b)

$$\mathbf{C}_f\mathbf{D}_f^{\,T} = \left[\begin{array}{cccc|ccc} 0 & -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & -1 & 0 & -1 \end{array}\right] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is a special case of the following general result: *for an oriented graph with n vertices and m edges, the matrix product* $\mathbf{C}_f\mathbf{D}_f^{\,T} = \mathbf{0}$. (that is, the zero matrix with $m - n + 1$ rows and $n - 1$ columns).

Solution 2.10



(a) (b)

For graph (a), the chords are *AD*, *AC* and *CD*, and the fundamental cycles are *ADBA*, *ACBA* and *BCDB*.

The fundamental cycle equations are

$$v_1 - v_5 - v_2 = 0$$
$$v_3 - v_4 - v_2 = 0$$
$$v_4 + v_6 - v_5 = 0.$$

For graph (a), the branches are *AB*, *BC* and *BD*, and the fundamental cutsets are {*AD*, *AB*, *AC*}, {*AC*, *BC*, *CD*} and {*AD*, *BD*, *CD*}.

The fundamental cutset equations are

$$i_1 + i_2 + i_3 = 0$$
$$i_3 + i_4 - i_6 = 0$$
$$i_1 + i_5 + i_6 = 0.$$

For graph (b), the chords are *WY*, *WX* and *XY*, and the fundamental cycles are *YZWY*, *ZWXZ* and *YZXY*.

The fundamental cycle equations are

$$v_1 + v_3 + v_2 = 0$$
$$v_3 + v_4 - v_6 = 0$$
$$v_1 + v_6 + v_5 = 0.$$

For graph (b), the branches are *YZ*, *ZW* and *ZX*, and the fundamental cutsets are {*YZ*, *WY*, *XY*}, {*WY*, *ZW*, *WX*} and {*WX*, *XY*, *ZX*}.

The fundamental cutset equations are

$$i_1 - i_2 - i_5 = 0$$
$$i_3 - i_2 - i_4 = 0$$
$$i_4 - i_5 + i_6 = 0.$$

Graph (b) is the dual of graph (a). See page 20.

Notice that the *thick* edges in graph (a) form a spanning tree; they correspond to the *thin* edges in graph (b); the *thin* edges in graph (a) correspond to the *thick* edges in graph (b), which form a spanning tree.

The fundamental cycle equations for graph (a) correspond to the fundamental cutset equations for graph (b), and the fundamental cutset equations for graph (a) correspond to the fundamental cycle equations for graph (b). (Interchange all the vs and is).

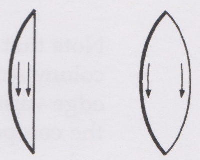
It can be shown that these results hold generally for the dual of a planar network.

Solution 3.1

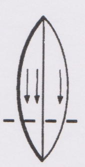
This example is similar to Example 3.2. The matrix equation is

$$\underset{\substack{\uparrow\\ \mathbf{T}}}{\left[\begin{array}{ccccccc:ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & -R_2 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -R_3 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -R_4 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -R_5 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -R_6 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -R_7 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]}
\underset{\substack{\uparrow\\ \left[\begin{array}{c}\mathbf{v}\\ \hdashline \mathbf{i}\end{array}\right]}}{\left[\begin{array}{c}
v_2\\ v_3\\ v_4\\ v_5\\ v_6\\ v_7\\ v_1\\ \hdashline i_2\\ i_3\\ i_4\\ i_5\\ i_6\\ i_7\\ i_1
\end{array}\right]}
=
\underset{\substack{\uparrow\\ \mathbf{f}}}{\begin{bmatrix}
0\\ 0\\ 0\\ 0\\ 0\\ 0\\ -V
\end{bmatrix}}$$

Solution 3.2

fundamental cycles: 

$$\mathbf{C}_f = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{c:cc} -1 & 1 & 0\\ -1 & 0 & 1 \end{array}\right] \end{array}$$

fundamental cutset: 

$$\mathbf{D}_f = \left[\begin{array}{c:cc} 1 & 1 & 1 \end{array}\right]$$

Combining these matrices with the matrix on page 44, and re-ordering the variables to agree with the matrices $\mathbf{C}_f$ and $\mathbf{D}_f$, we get

$$\underset{\substack{\uparrow\\ \mathbf{H}}}{\left[\begin{array}{ccc:ccc}
0 & 0 & 0 & 1 & 0 & 0\\
0 & 1 & 0 & 0 & -R_2 & 0\\
0 & 0 & 1 & 0 & 0 & -R_3\\ \hdashline
-1 & 1 & 0 & 0 & 0 & 0\\
-1 & 0 & 1 & 0 & 0 & 0\\ \hdashline
0 & 0 & 0 & 1 & 1 & 1
\end{array}\right]}
\underset{\substack{\uparrow\\ \mathbf{x}}}{\begin{bmatrix} v_1\\ v_2\\ v_3\\ i_1\\ i_2\\ i_3 \end{bmatrix}}
=
\underset{\substack{\uparrow\\ \mathbf{y}}}{\left[\begin{array}{c} I\\ 0\\ 0\\ \hdashline 0\\ 0\\ \hdashline 0 \end{array}\right]}$$

Notice that we have made the current source the first component in the column vector. It does not matter where the *independent* sources appear, as long as the network voltages and currents appear in the same order in the component, cycle and cutset equations.

Solution 3.3

$$
\left[\begin{array}{ccccccc|ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & -R_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -R_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -R_6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -R_5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -R_4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -R_7 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1
\end{array}\right]
\begin{bmatrix} v_2 \\ v_3 \\ v_6 \\ v_5 \\ v_4 \\ v_7 \\ v_1 \\ i_2 \\ i_3 \\ i_6 \\ i_5 \\ i_4 \\ i_7 \\ i_1 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -V \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Solution 3.4

(a)

$$
\left[\begin{array}{cccccc|cc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1/L & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -R_4 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/C & 0 & 0 & 0
\end{array}\right]
\begin{bmatrix} v_1 \\ v_3 \\ v_4 \\ i_1 \\ i_2 \\ i_4 \\ \hline v_2 \\ i_3 \end{bmatrix}
=
\begin{bmatrix} k \\ 0 \\ 0 \\ 0 \end{bmatrix}
+
\left[\begin{array}{cccccc|cc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
\begin{bmatrix} \dot{v}_1 \\ \dot{v}_3 \\ \dot{v}_4 \\ \dot{i}_1 \\ \dot{i}_2 \\ \dot{i}_4 \\ \hline \dot{v}_2 \\ \dot{i}_3 \end{bmatrix}
$$

Note that the variables whose derivatives occur in the component equations (that is, v_2 and i_3) appear last in the column vectors.

(b) The fundamental cycle and cutset matrices can be rewritten as follows:

$$
\left[\begin{array}{cccccc|cc}
1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 1
\end{array}\right]
\begin{bmatrix} v_1 \\ v_3 \\ v_4 \\ i_1 \\ i_2 \\ i_4 \\ \hline v_2 \\ i_3 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Note that we have rearranged the columns of the matrix so as to put the edge variables in the same order as in the component equation matrix.

Combining this with the component-equation matrix, we obtain the **H**-matrix equation $\mathbf{Hx} = \mathbf{y} + \mathbf{K\dot{x}}$, as follows:

$$
\left[\begin{array}{cccccc|cc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1/L & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -R_4 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/C & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 1
\end{array}\right]
\begin{bmatrix} v_1 \\ v_3 \\ v_4 \\ i_1 \\ i_2 \\ i_4 \\ \hline v_2 \\ i_3 \end{bmatrix}
=
\begin{bmatrix} k \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \end{bmatrix}
+
\left[\begin{array}{cccccc|cc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\begin{bmatrix} \dot{v}_1 \\ \dot{v}_3 \\ \dot{v}_4 \\ \dot{i}_1 \\ \dot{i}_2 \\ \dot{i}_4 \\ \hline \dot{v}_2 \\ \dot{i}_3 \end{bmatrix}
$$

(c) To find the state equations we start with rows 5 and 8 and perform row operations in such a way that the first six entries in these rows are all zero.

First state equation: Take row 5, and subtract row 1 (to eliminate the 1 in column 1) and $L \times$ row 2 (to eliminate the 1 in column 2). Performing the same operations on the right-hand side of the equation yields the state equation

$$v_2 = -k - L\dot{i}_3$$

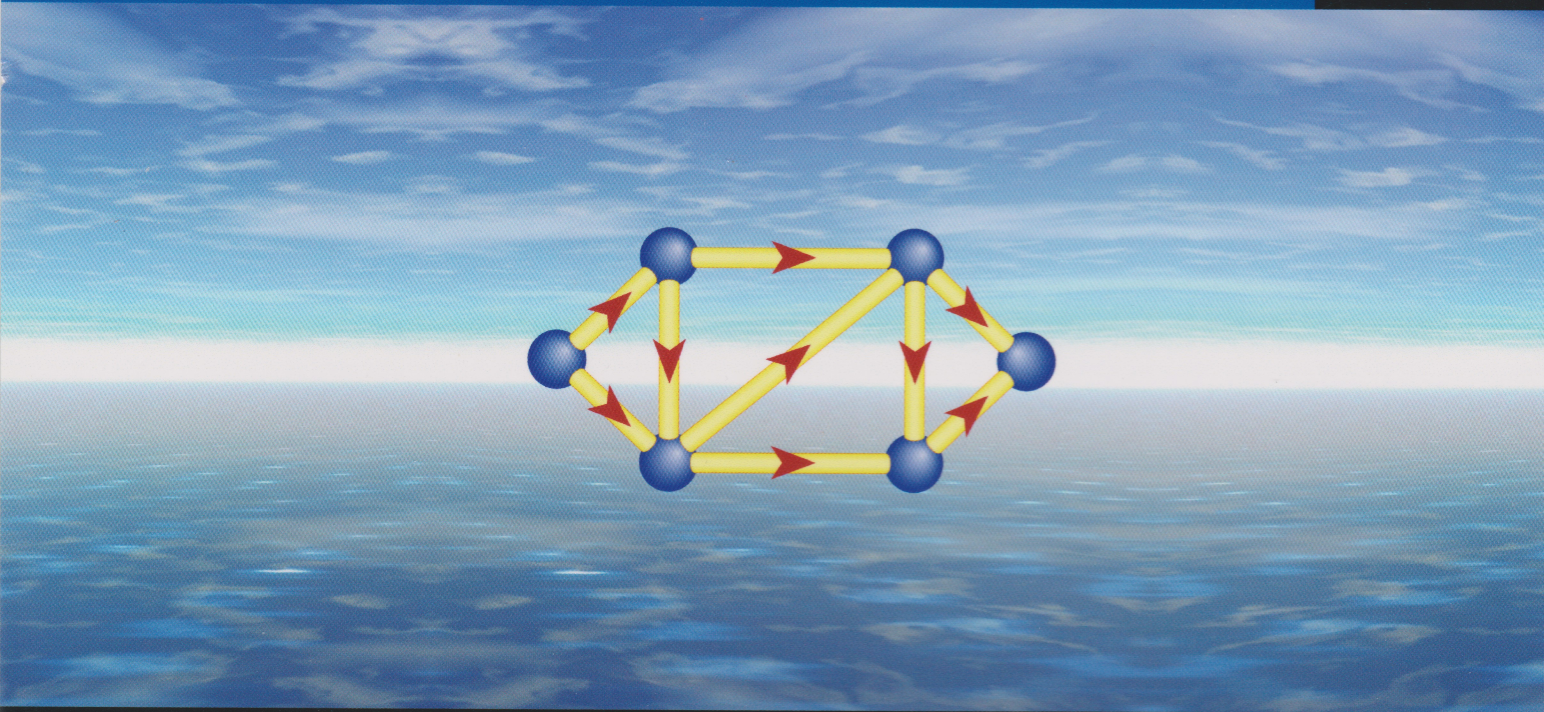
Second state equation: We can eliminate the entries in columns 5 and 6 of row 8 by taking

$$(\text{row } 8) + (C \times \text{row } 4) + ((1/R_4) \times \text{row } 3) - ((1/R_4) \times \text{row } 6) - ((L/R_4) \times \text{row } 2)$$

This yields the state equation

$$i_3 = C\dot{v}_2 - \frac{L}{R_4}\dot{i}_3.$$

Index



MT365 Graphs, networks and design

The Open University

MT365 Networks 4

ISBN 0 7492 2228 X